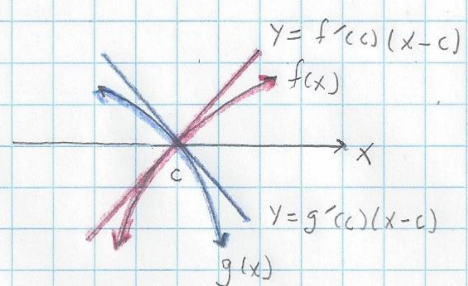
$\frac{0}{0}$ case



$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$, but, in a small neighborhood of $x=c$, $f(x)$ looks like its tangent line, and $g(x)$ looks like its tangent line $\Rightarrow$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(c)\cancel{(x-c)}}{g'(c)\cancel{(x-c)}} = \lim_{x\to c} \frac{f'(c)}{g'(c)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then $\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}}$

L'Hôpital's Rule

$\frac{\infty}{\infty}$ case

$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ Let $f(x) = \frac{1}{F(x)}$ so that as $x \to c$, $f(x) \to \infty$ and $F(x) \to 0$. Also, let $g(x) = \frac{1}{G(x)}$ so that as $x \to c$, $g(x) \to \infty$ and $G(x) \to 0$.

$$\frac{d}{dx}\left(\frac{1}{F}\right) = -\frac{1}{F^2} \cdot \frac{dF}{dx} = -\frac{F'(x)}{F^2(x)} \qquad \frac{d}{dx}\left(\frac{1}{G}\right) = -\frac{G'(x)}{G^2(x)},$$

$$\frac{f'(x)}{g'(x)} = \frac{\frac{d}{dx}\left(\frac{1}{F}\right)}{\frac{d}{dx}\left(\frac{1}{G}\right)} = \frac{-\frac{F'(x)}{F^2(x)}}{-\frac{G'(x)}{G^2(x)}} = \frac{F'(x)}{G'(x)} \cdot \frac{G^2(x)}{F^2(x)}, \quad \frac{G}{F} = \frac{\frac{1}{g}}{\frac{1}{f}} = \frac{f}{g} \Rightarrow$$

$$\frac{f'(x)}{g'(x)} = \frac{F'(x)}{G'(x)} \cdot \frac{G(x)}{F(x)} \cdot \frac{f(x)}{g(x)},$$

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = \left[\lim_{x\to c} \frac{F'(x)}{G'(x)}\right]\left[\lim_{x\to c} \frac{G(x)}{F(x)}\right]\left[\lim_{x\to c} \frac{f(x)}{g(x)}\right] =$$

$\frac{0}{0}$ case of L'Hôpital's Rule

$$= \left[\lim_{x\to c} \frac{\cancel{F'(x)}}{\cancel{G'(x)}}\right]\left[\lim_{x\to c} \frac{\cancel{G'(x)}}{\cancel{F'(x)}}\right]\left[\lim_{x\to c} \frac{f(x)}{g(x)}\right] = \lim_{x\to c} \frac{f(x)}{g(x)}$$

9.2. L'Hôpital's Rule

If $\lim\limits_{x\to c} f(x) = \infty$ and $\lim\limits_{x\to c} g(x) = \infty$, then $\boxed{\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}}$

Example #1. Calculate the limits using L'Hôpital's Rule.

(a) $\lim\limits_{x\to 0} \dfrac{(4+x)^3-64}{x}$ (b) $\lim\limits_{x\to 0^+} x\csc x$

(c) $\lim\limits_{x\to\infty} \dfrac{\ln x}{3^x}$ (d) $\lim\limits_{x\to 0} \dfrac{\cos x-1}{x}$

SOLUTION:

(a) $\lim\limits_{x\to 0} \dfrac{(4+x)^3-64}{x} = \dfrac{0}{0} = \lim\limits_{x\to 0} \dfrac{3(4+x)^2}{1} = 3\cdot 16 = 48$ ◄

(b) $\lim\limits_{x\to 0^+} x\csc x = 0\cdot\infty = \lim\limits_{x\to 0^+} \dfrac{x}{\sin x} = \dfrac{0}{0} = \lim\limits_{x\to 0^+} \dfrac{1}{\cos x} = \dfrac{1}{1} = 1$ ◄

(c) $\lim\limits_{x\to\infty} \dfrac{\ln x}{3^x} = \dfrac{\infty}{\infty} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{(\ln 3)3^x} = \lim\limits_{x\to\infty} \dfrac{1}{x(\ln 3)3^x} = \dfrac{1}{(\ln 3)\infty\cdot\infty} = 0$ ◄

(d) $\lim\limits_{x\to 0} \dfrac{\cos x-1}{x} = \dfrac{0}{0} = \lim\limits_{x\to 0} \dfrac{-\sin x}{1} = \dfrac{0}{1} = 0$ ◄

■ Use of logarithms

If $\lim\limits_{x\to c} \ln f(x) = L$, then $\lim\limits_{x\to c} f(x) = e^L$

Example #2. Calculate $\lim\limits_{x\to 0^+} x^x$.

SOLUTION: $\lim\limits_{x\to 0^+} x^x = 0^0$. $\lim\limits_{x\to 0^+} \ln x^x = \lim\limits_{x\to 0^+} x\ln x = 0\cdot -\infty =$

$= \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \dfrac{-\infty}{\infty} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} (-x) = 0 \Rightarrow \lim\limits_{x\to 0^+} x^x = e^0 = 1$ ◄

9.2. L'Hôpital's Rule

CLASS WORK

(1) Use L'Hôpital's Rule to calculate the limits

(a) $\lim\limits_{x \to 0} \dfrac{(2+x)^4 - 16}{x}$ (b) $\lim\limits_{x \to 0} \sqrt{x}\log_3 x$

(c) $\lim\limits_{x \to \infty} \dfrac{\sqrt{x}}{\ln x}$ (d) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$

(2) Use the natural logarithm to calculate $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

SOLUTIONS

(1) (a) $\lim\limits_{x \to 0} \dfrac{(2+x)^4 - 16}{x} = \dfrac{0}{0} = \lim\limits_{x \to 0} \dfrac{4(2+x)^3}{1} = 4 \cdot 2^3 = 32$

(b) $\lim\limits_{x \to 0} \sqrt{x}\log_3 x = 0 \cdot -\infty = \lim\limits_{x \to 0} \dfrac{\log_3 x}{\frac{1}{\sqrt{x}}} = \dfrac{-\infty}{\infty} = \lim\limits_{x \to 0} \dfrac{\frac{1}{(\ln 3)x}}{-\frac{1}{2\sqrt{x^3}}} = \lim\limits_{x \to 0} \dfrac{-2\sqrt{x}}{\ln 3} = 0$

(c) $\lim\limits_{x \to \infty} \dfrac{\sqrt{x}}{\ln x} = \dfrac{\infty}{\infty} = \lim\limits_{x \to \infty} \dfrac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim\limits_{x \to \infty} \dfrac{\sqrt{x}}{2} = \infty$

(d) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = \dfrac{0}{0} = \lim\limits_{x \to 0} \dfrac{e^x}{1} = \dfrac{1}{1} = 1$

(2) $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = 1^\infty$. $\lim\limits_{x \to \infty} \ln\left(1 + \frac{1}{x}\right)^x = \lim\limits_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right) = \infty \cdot 0 =$

$= \lim\limits_{x \to \infty} \dfrac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \dfrac{0}{0} = \lim\limits_{x \to \infty} \dfrac{\frac{1}{1+\frac{1}{x}} \cdot -\frac{1}{x^2}}{-\frac{1}{x^2}} = \lim\limits_{x \to \infty} \dfrac{1}{1 + \frac{1}{x}} = \dfrac{1}{1+0} = 1 \Rightarrow$

$\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e^1 = e$