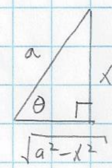
Supplemental

7.2. Trigonometric Substitutions

(1) $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}$

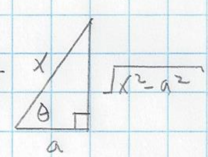


$\sin\theta = \frac{x}{a}$, $x = a\sin\theta$, $\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$, $\sqrt{a^2-x^2} = a\cos\theta$,

$\frac{dx}{d\theta} = a\cos\theta$, $dx = a\cos\theta\, d\theta$, $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{\cancel{a\cos\theta}\, d\theta}{\cancel{a\cos\theta}} = \int d\theta = \theta + k = \sin^{-1}\left(\frac{x}{a}\right) + k$$

(2) $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$



$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$, $\sqrt{x^2-a^2} = a\tan\theta$,

$\cos\theta = \frac{a}{x}$, $\sec\theta = \frac{x}{a}$, $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\, d\theta$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{\cancel{a}\sec\theta\,\cancel{\tan\theta}\, d\theta}{\cancel{a}\,\cancel{\tan\theta}} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + k =$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + k = \ln\left|x + \sqrt{x^2-a^2}\right| \underbrace{- \ln|a|} + k =$$

can leave off because it is a constant

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + k$$

Supplemental

7.3. Polynomials times Functions whose Derivatives Repeat

(3) $f(x) = Ax^2e^x + Bxe^x + Ce^x$, $f'(x) = A(2xe^x + x^2e^x) + B(e^x + xe^x) + Ce^x =$

$= Ax^2e^x + (2A+B)xe^x + (B+C)e^x = x^2e^x \Rightarrow A = 1,\ B = -2,\ C = 2 \Rightarrow$

$\int x^2e^x\,dx = \int f'(x)\,dx = f(x) + k = x^2e^x - 2xe^x + 2e^x + k$

(4) $f(x) = Ax^2\sin x + Bx^2\cos x + Cx\sin x + Dx\cos x + E\sin x + F\cos x$,

$f'(x) = A(2x\sin x + x^2\cos x) + B(2x\cos x - x^2\sin x) + C(\sin x + x\cos x) +$

$+ D(\cos x - x\sin x) + E\cos x - F\sin x =$

$= Ax^2\cos x - Bx^2\sin x + (2A-D)x\sin x + (2B+C)x\cos x + (C-F)\sin x + (D+E)\cos x =$

$= x^2\sin x \Rightarrow A = 0,\ B = -1,\ C = 2,\ D = 0,\ E = 0,\ F = 2 \Rightarrow$

$\int x^2\sin x\,dx = \int f'(x)\,dx = f(x) + k = -x^2\cos x + 2x\sin x + 2\cos x + k$

(5) (a) $\dfrac{d^{369} e^{-x}}{dx^{369}} = \dfrac{d}{dx} \dfrac{d^{368} e^{-x}}{dx^{368}} = \dfrac{d}{dx} e^{-x} = -e^{-x}$

(b) $\dfrac{d^{369} \sinh x}{dx^{369}} = \dfrac{d}{dx} \dfrac{d^{368} \sinh x}{dx^{368}} = \dfrac{d}{dx} \sinh x = \cosh x$

(c) $\dfrac{d^{369} \cos x}{dx^{369}} = \dfrac{d}{dx} \dfrac{d^{368} \cos x}{dx^{368}} = \dfrac{d}{dx} \cos x = -\sin x$

pg. 364 7.4. Exponential Growth and Decay

(24) $p = p_0 e^{kt}$, $\begin{matrix} 40,000 = p_0 e^{5k} \\ 10,000 = p_0 e^{3k} \end{matrix}$ $(\div) \Rightarrow 4 = e^{2k}$, $2k = \ln 4$, $k = \frac{1}{2} \ln 4 = \ln 2$, $k = 0.693$

$10,000 = p_0 e^{3k}$, $p_0 = \dfrac{10,000}{e^{3k}} = 1250$

(35) $c = c_0 e^{-kt}$, $0.445\, \cancel{c_0} = \cancel{c_0} e^{-kt}$, $-kt = \ln 0.445$, $t = -\frac{1}{k} \ln 0.445 = 6693$ yrs

Supplemental 7.4. Newton's Law of Cooling (or Heating)

(6) $\dfrac{dy}{dx} = 3 - 2y$, $\dfrac{dy}{3-2y} = dx$, $u = 3 - 2y$, $\dfrac{du}{dy} = -2$, $du = -2dy$, $dy = -\dfrac{du}{2}$,

$-\dfrac{du}{2} \cdot \dfrac{1}{u} = dx$, $\dfrac{du}{u} = -2dx$, $y(0) = 5 \Rightarrow u_0 = -7$, $\displaystyle\int_{u_0}^{u} \frac{dz}{z} = -2 \int_0^x dt$,

$\left[\ln z\right]_{-7}^{u} = \left[\ln z\right]_{-7}^{3-2y} = -2 \left[t\right]_0^x$, $\ln\left(\dfrac{3-2y}{-7}\right) = -2x$, $\dfrac{3-2y}{-7} = e^{-2x}$,

$3 - 2y = -7e^{-2x}$, $2y - 3 = 7e^{-2x}$, $2y = 3 + 7e^{-2x}$, $y = 1.5 + 3.5e^{-2x}$

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(30) $T = T_\infty + (T_0 - T_\infty) e^{-kt}$, $T = 65 + (T_0 - 65) e^{-kt}$

$\begin{matrix} 35 = 65 + (T_0 - 65) e^{-10k}, & -30 = (T_0 - 65) e^{-10k} \\ 50 = 65 + (T_0 - 65) e^{-20k}, & -15 = (T_0 - 65) e^{-20k} \end{matrix}$ $(\div) \Rightarrow$

$2 = e^{10k}$, $10k = \ln 2$, $k = \frac{1}{10} \ln 2 = 0.069315$/min

$-15 e^{20k} = T_0 - 65$, $T_0 = 65 - 15e^{20k} = 5°F$

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(31) $T = T_\infty + (T_0 - T_\infty)e^{-kt}$, $T = 20 + (90-20)e^{-kt} = 20 + 70e^{-kt}$

(a) $60 = 20 + 70e^{-10k}$, $40 = 70e^{-10k}$, $e^{-10k} = \frac{4}{7}$, $-10k = \ln\left(\frac{4}{7}\right)$,

$k = -\frac{1}{10}\ln\left(\frac{4}{7}\right) = 0.055962/\text{min}$, $35 = 20 + 70e^{-kt}$, $15 = 70e^{-kt}$,

$\frac{3}{14} = e^{-kt}$, $-kt = \ln\left(\frac{3}{14}\right)$, $t = -\frac{1}{k}\ln\left(\frac{3}{14}\right) = 27.527$ min $(-10) \Rightarrow$

17.527 min $= 17{:}32$ ◀

(b) $T = T_\infty + (T_0 - T_\infty)e^{-kt}$, $T = -15 + (90 - -15)e^{-kt} = -15 + 105e^{-kt}$,

$35 = -15 + 105e^{-kt}$, $50 = 105e^{-kt}$, $e^{-kt} = \frac{10}{21}$, $-kt = \ln\left(\frac{10}{21}\right)$,

$t = -\frac{1}{k}\ln\left(\frac{10}{21}\right) = 13.258$ min $= 13{:}15$ ◀

pg. 458 9.2. L'Hôpital's Rule

(2) $\lim_{x\to 0} \frac{\sin 5x}{x} = \frac{0}{0} = \lim_{x\to 0} \frac{5\cos 5x}{1} = \frac{5\cdot 1}{1} = 5$ ◀

(3) $\lim_{x\to 2} \frac{\sqrt{2+x} - 2}{x-2} = \frac{0}{0} = \lim_{x\to 2} \frac{\frac{1}{2\sqrt{2+x}}}{1} = \lim_{x\to 2} \frac{1}{2\sqrt{2+x}} = \frac{1}{2\cdot\sqrt{4}} = \frac{1}{4}$ ◀

(7) $\lim_{t\to 0} \frac{\cos t - 1}{e^t - t - 1} = \frac{0}{0} = \lim_{t\to 0} \frac{-\sin t}{e^t - 1} = \frac{0}{0} = \lim_{t\to 0} \frac{-\cos t}{e^t} = \frac{-1}{1} = -1$ ◀

(15) $\lim_{x\to\infty} \frac{\ln(x+1)}{\log_2 x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{(\ln 2)x}} = \lim_{x\to\infty} \frac{(\ln 2)x}{x+1} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{\ln 2}{2} = \ln\sqrt{2}$ ◀

(19) $\lim_{x\to 0^+} (\csc x - \cot x + \cos x) = \lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} + \frac{\cos x \sin x}{\sin x}\right) =$

$= \lim_{x\to 0^+} \left(\frac{1 - \cos x + \sin x \cos x}{\sin x}\right) = \frac{0}{0} = \lim_{x\to 0^+} \left(\frac{\sin x + \cos x \cos x + \sin x \cdot -\sin x}{\sin x}\right) =$

$= \lim_{x\to 0^+} \left(\frac{\sin x + \cos^2 x - \sin^2 x}{\sin x}\right) = \frac{1}{0} = \infty$ ◀

HW#12

(21) $\lim_{x\to 0} (e^x + x)^{1/x} = L = 1^{\infty}$, $\lim_{x\to 0} \ln(e^x+x)^{1/x} = \lim_{x\to 0} \frac{\ln(e^x+x)}{x} = \frac{0}{0} =$

$= \lim_{x\to 0} \frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1} = \lim_{x\to 0} \frac{e^x+1}{e^x+x} = \frac{1+1}{1} = 2 = \ln L \Rightarrow \lim_{x\to 0} (e^x+x)^{1/x} = e^2$ ◂

(24) $\lim_{x\to 0^+} (\sin x)^x = 0^0 = L$, $\lim_{x\to 0^+} \ln\left[(\sin x)^x\right] = \lim_{x\to 0^+} x\ln(\sin x) = 0\cdot -\infty =$

$= \lim_{x\to 0^+} \frac{\ln(\sin x)}{\frac{1}{x}} = \frac{-\infty}{\infty} = \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2\cos x}{\sin x} = \frac{0}{0} =$

$= \lim_{x\to 0^+} \frac{-(2x\cos x + x^2\cdot -\sin x)}{\cos x} = \frac{0}{1} = 0 = \ln L \Rightarrow \lim_{x\to 0^+} (\sin x)^x = e^0 = 1$ ◂