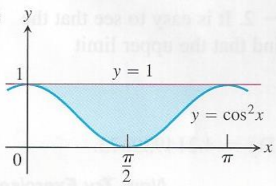
Section 8.2 Exercises

In Exercises 1–6, find the area of the shaded region analytically.

1.



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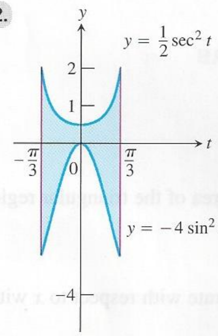

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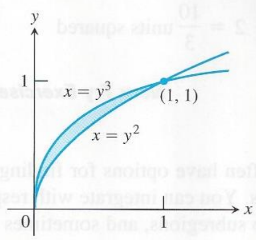

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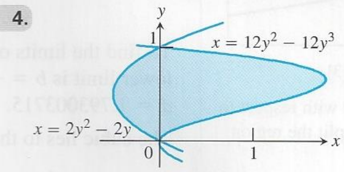

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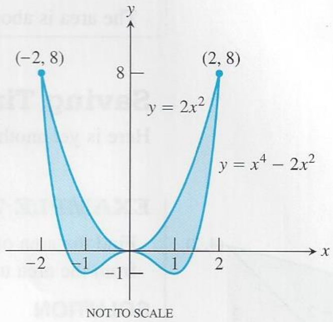

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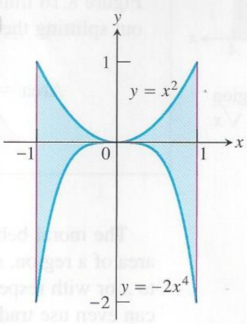
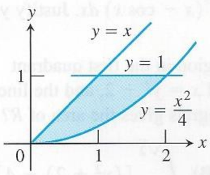


In Exercises 7 and 8, use a calculator to find the area of the region enclosed by the graphs of the two functions.

7. $y = \sin x,\ y = 1 - x^2$ **8.** $y = \cos(2x),\ y = x^2 - 2$

In Exercises 9 and 10, find the area of the shaded region analytically.

9.



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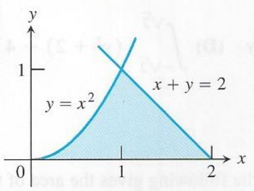
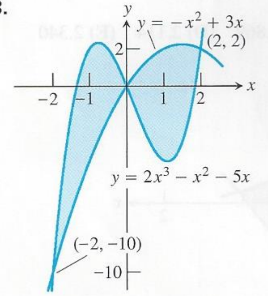


In Exercises 11 and 12, find the area enclosed by the graphs of the two curves by integrating with respect to y.

11. $y^2 = x + 1,\ y^2 = 3 - x$ **12.** $y^2 = x + 3,\ y = 2x$

In Exercises 13 and 14, find the total shaded area.

13.



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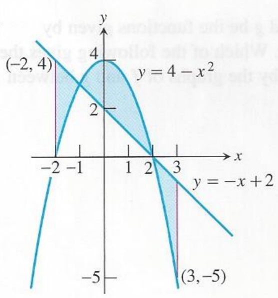


In Exercises 15–34, find the area of the regions enclosed by the lines and curves.

15. $y = x^2 - 2$ and $y = 2$

16. $y = 2x - x^2$ and $y = -3$

17. $y = 7 - 2x^2$ and $y = x^2 + 4$

18. $y = x^4 - 4x^2 + 4$ and $y = x^2$

19. $y = x\sqrt{a^2 - x^2},\ a > 0,$ and $y = 0$

20. $y = \sqrt{|x|}$ and $5y = x + 6$
(How many intersection points are there?)

21. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

22. $x = y^2$ and $x = y + 2$

23. $y^2 - 4x = 4$ and $4x - y = 16$

24. $x - y^2 = 0$ and $x + 2y^2 = 3$

25. $x + y^2 = 0$ and $x + 3y^2 = 2$

26. $4x^2 + y = 4$ and $x^4 - y = 1$

27. $x + y^2 = 3$ and $4x + y^2 = 0$

28. $y = 2\sin x$ and $y = \sin 2x,\quad 0 \le x \le \pi$

29. $y = 8\cos x$ and $y = \sec^2 x, -\pi/3 \le x \le \pi/3$

30. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

31. $y = \sin(\pi x/2)$ and $y = x$

32. $y = \sec^2 x,\quad y = \tan^2 x,\quad x = -\pi/4,\quad x = \pi/4$

33. $x = \tan^2 y$ and $x = -\tan^2 y,\quad -\pi/4 \le y \le \pi/4$

34. $x = 3\sin y\sqrt{\cos y}$ and $x = 0,\quad 0 \le y \le \pi/2$

In Exercises 35 and 36, find the area of the region by subtracting the area of a triangular region from the area of a larger region.

35. The region on or above the x-axis bounded by the curves $y^2 = x + 3$ and $y = 2x$

36. The region on or above the x-axis bounded by the curves $y = 4 - x^2$ and $y = 3x$

37. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

38. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

39. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

40. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **(a)** x, **(b)** y.

41. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

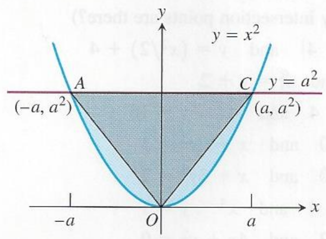
(a) Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

(b) Find c by integrating with respect to y. (This puts c in the limits of integration.)

(c) Find c by integrating with respect to x. (This puts c into the integrand as well.)

42. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

43. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

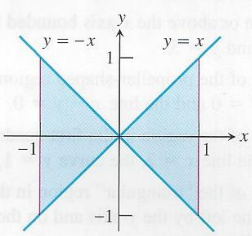


44. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

45. Writing to Learn Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

i. $\displaystyle\int_{-1}^{1} (x - (-x))\,dx = \int_{-1}^{1} 2x\,dx$

ii. $\displaystyle\int_{-1}^{1} (-x - (x))\,dx = \int_{-1}^{1} -2x\,dx$



46. Writing to Learn Is the following statement true, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

Give reasons for your answer.

47. Find the area of the propeller-shaped region enclosed between the graphs of

$$y = \frac{2x}{x^2 + 1} \quad \text{and} \quad y = x^3.$$

48. Find the area of the propeller-shaped region enclosed between the graphs of $y = \sin x$ and $y = x^3$.

49. Find the positive value of k such that the area of the region enclosed between the graph of $y = k \cos x$ and the graph of $y = kx^2$ is 2.

Standardized Test Questions

50. True or False The area of the region enclosed by the graph of $y = x^2 + 1$ and the line $y = 10$ is 36. Justify your answer.

51. True or False The area of the region in the first quadrant enclosed by the graphs of $y = \cos x$, $y = x$, and the y-axis is given by the definite integral $\int_0^{0.739} (x - \cos x)\,dx$. Justify your answer.

52. Multiple Choice Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = y^2 + 2$, and the line $x = 4$. Which of the following integrals gives the area of R?

(A) $\displaystyle\int_0^{\sqrt{2}} [4 - (y^2 + 2)]\,dy$ (B) $\displaystyle\int_0^{\sqrt{2}} [(y^2 + 2) - 4]\,dy$

(C) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} [4 - (y^2 + 2)]\,dy$ (D) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} [(y^2 + 2) - 4]\,dy$

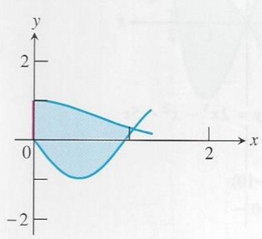
(E) $\displaystyle\int_2^4 [4 - (y^2 + 2)]\,dy$

53. Multiple Choice Which of the following gives the area of the region between the graphs of $y = x^2$ and $y = -x$ from $x = 0$ to $x = 3$?

(A) 2 (B) $9/2$ (C) $13/2$ (D) 13 (E) $27/2$

54. Multiple Choice Let R be the shaded region enclosed by the graphs of $y = e^{-x^2}$, $y = -\sin(3x)$, and the y-axis as shown in the figure below. Which of the following gives the approximate area of the region R?

(A) 1.139 (B) 1.445 (C) 1.869 (D) 2.114 (E) 2.340



55. Multiple Choice Let f and g be the functions given by $f(x) = e^x$ and $g(x) = 1/x$. Which of the following gives the area of the region enclosed by the graphs of f and g between $x = 1$ and $x = 2$?

(A) $e^2 - e - \ln 2$

(B) $\ln 2 - e^2 + e$

(C) $e^2 - \dfrac{1}{2}$

(D) $e^2 - e - \dfrac{1}{2}$

(E) $\dfrac{1}{e} - \ln 2$

Exploration

56. Group Activity ***Area of Ellipse***

An ellipse with major axis of length $2a$ and minor axis of length $2b$ can be coordinatized with its center at the origin and its major axis horizontal, in which case it is defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(a) Find the equations that define the upper and lower semi-ellipses as functions of x.

(b) Write an integral expression that gives the area of the ellipse.

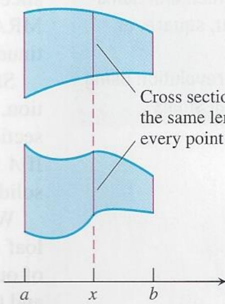
(c) With your group, use NINT to find the areas of ellipses for various lengths of a and b.

(d) There is a simple formula for the area of an ellipse with major axis of length $2a$ and minor axis of length $2b$. Can you tell what it is from the areas you and your group have found?

(e) Work with your group to write a *proof* of this area formula by showing that it is the exact value of the integral expression in part (b).

Extending the Ideas

57. ***Cavalieri's Theorem*** Bonaventura Cavalieri (1598–1647) discovered that if two plane regions can be arranged to lie over the same interval of the x-axis in such a way that they have identical vertical cross sections at every point (see figure), then the regions have the same area. Show that this theorem is true.



58. Find the area of the region enclosed by the curves

$$y = \frac{x}{x^2 + 1} \quad \text{and} \quad y = mx, \quad 0 < m < 1.$$