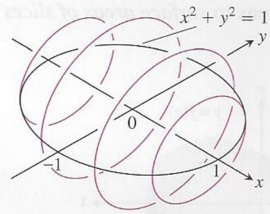
Section 8.3 Exercises

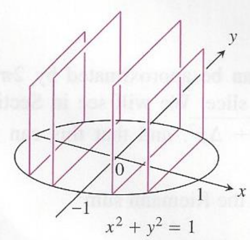
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross sections of the solid that are perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

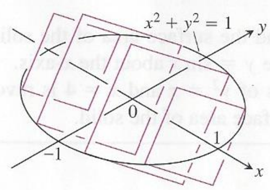
(a) The cross sections are circular disks with diameters in the xy-plane.



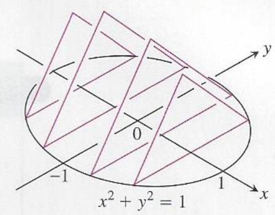
(b) The cross sections are squares with bases in the xy-plane.



(c) The cross sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

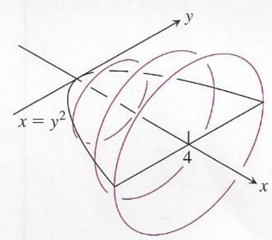


(d) The cross sections are equilateral triangles with bases in the xy-plane.

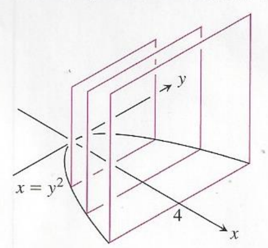


2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

(a) The cross sections are circular disks with diameters in the xy-plane.



(b) The cross sections are squares with bases in the xy-plane.



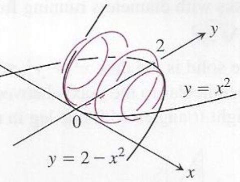
(c) The cross sections are squares with diagonals in the xy-plane.

(d) The cross sections are equilateral triangles with bases in the xy-plane.

In Exercises 3–6, find the volume of the solid analytically.

3. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from $y = -\sqrt{x}$ to $y = \sqrt{x}$.

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

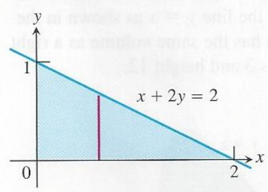


5. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

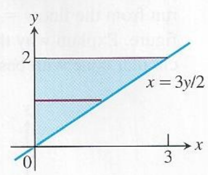
6. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

In Exercises 7–10, find the volume of the solid generated by revolving the shaded region about the given axis.

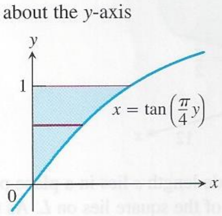
7. about the x-axis



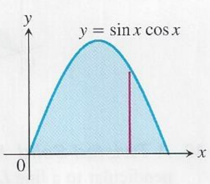
8. about the y-axis



9. about the y-axis



10. about the x-axis



In Exercises 11–20, find the volume of the solid generated by revolving the region bounded by the lines and curves about the x-axis.

11. $y = x^2, \quad y = 0, \quad x = 2$

12. $y = x^3, \quad y = 0, \quad x = 2$

13. $y = \sqrt{9 - x^2}, \quad y = 0$

14. $y = x - x^2, \quad y = 0$

15. $y = x, \quad y = 1, \quad x = 0$

16. $y = 2x, \quad y = x, \quad x = 1$

17. $y = x^2 + 1, y = x + 3$

18. $y = 4 - x^2, \quad y = 2 - x$

19. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

20. $y = -\sqrt{x}, \quad y = -2, \quad x = 0$

In Exercises 21–24, find the volume of the solid generated by revolving the region about the given line.

21. the region bounded by $y = x^2$, $y = 0$, and $x = 2$ about the line $x = 2$

22. the region bounded by $y = \sin x$, $y = 0$, and $x = \pi/2$ about the line $x = \pi/2$

23. the region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$

24. the region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$

In Exercises 25–30, find the volume of the solid generated by revolving the region about the y-axis.

25. the region enclosed by $x = \sqrt{5}y^2, \quad x = 0, \quad y = -1, \quad y = 1$

26. the region enclosed by $x = y^{3/2}, \quad x = 0, \quad y = 2$

27. the region enclosed by the triangle with vertices $(1, 0)$, $(2, 1)$, and $(1, 1)$

28. the region enclosed by the triangle with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$

29. the region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$

30. the region bounded above by the curve $y = \sqrt{x}$ and below by the line $y = x$

Group Activity In Exercises 31–34, find the volume of the solid described.

31. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

(a) the x-axis. **(b)** the y-axis.
(c) the line $y = 2$. **(d)** the line $x = 4$.

32. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

(a) the line $x = 1$. **(b)** the line $x = 2$.

33. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

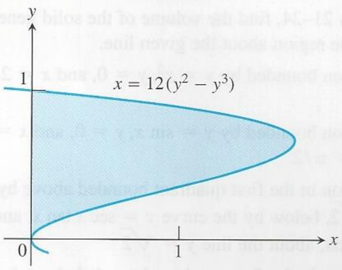
(a) the line $y = 1$. **(b)** the line $y = 2$.
(c) the line $y = -1$.

34. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about

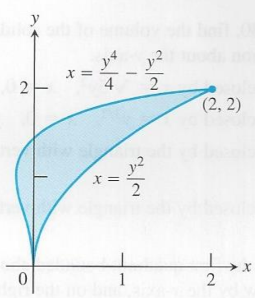
(a) the x-axis. **(b)** the y-axis.

In Exercises 35 and 36, use the cylindrical shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis.

35. **(a)** the x-axis **(b)** the line $y = 1$
(c) the line $y = 8/5$ **(d)** the line $y = -2/5$



36. **(a)** the x-axis **(b)** the line $y = 2$
(c) the line $y = 5$ **(d)** the line $y = -5/8$



In Exercises 37–40, use the cylindrical shell method to find the volume of the solid generated by revolving the region bounded by the curves about the y-axis.

37. $y = x, \quad y = -x/2, \quad x = 2$

38. $y = x^2, \quad y = 2 - x, \quad x = 0, \quad \text{for} \quad x \geq 0$

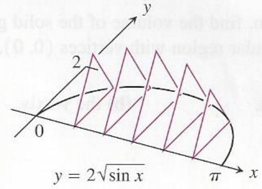
39. $y = \sqrt{x}, \quad y = 0, \quad x = 4$

40. $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$

In Exercises 41–44, find the volume of the solid analytically.

41. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross sections perpendicular to the x-axis are

(a) equilateral triangles with bases running from the x-axis to the curve as shown in the figure.



(b) squares with bases running from the x-axis to the curve.

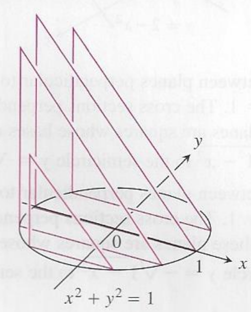
42. The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the x-axis are

(a) circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

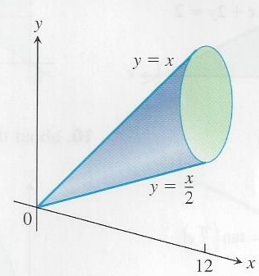
(b) squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.

43. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$.

44. The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



45. **Writing to Learn** A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 12$. The cross sections by planes perpendicular to the x-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



46. ***A Twisted Solid*** A square of side length s lies in a plane perpendicular to a line L. One vertex of the square lies on L. As this square moves a distance h along L, the square turns one revolution about L to generate a corkscrew-like column with square cross sections.

(a) Find the volume of the column.

(b) **Writing to Learn** What will the volume be if the square turns twice instead of once? Give reasons for your answer.

47. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = x^3$ and $y = 4x$ about

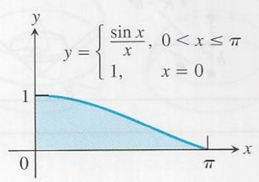
(a) the x-axis,

(b) the line $y = 8$.

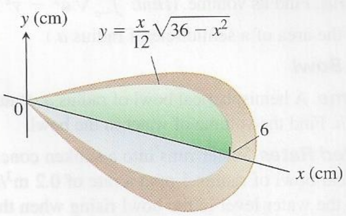
48. Find the volume of the solid generated by revolving the region bounded by $y = 2x - x^2$ and $y = x$ about
(a) the y-axis,
(b) the line $x = 1$.

49. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by (a) the washer method and (b) the cylindrical shell method.

50. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \le \pi \\ 1, & x = 0. \end{cases}$
(a) Show that $x\,f(x) = \sin x,\quad 0 \le x \le \pi$.
(b) Find the volume of the solid generated by revolving the shaded region about the y-axis.



51. ***Designing a Plumb Bob*** Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here.



(a) Find the plumb bob's volume.
(b) If you specify a brass that weighs 8.5 g/cm^3, how much will the plumb bob weigh to the nearest gram?

52. ***Volume of a Bowl*** A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y-axis.
(a) Find the volume of the bowl.
(b) If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?

53. ***The Classical Bead Problem*** A round hole is drilled through the center of a spherical solid of radius r. The resulting cylindrical hole has height 4 cm.
(a) What is the volume of the solid that remains?
(b) What is unusual about the answer?

54. **Writing to Learn** Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

55. ***Same Volume About Each Axis*** The region in the first quadrant enclosed between the graph of $y = ax - x^2$ and the x-axis generates the same volume whether it is revolved about the x-axis or the y-axis. Find the value of a.

56. *(Continuation of Exploration 2)* Let $x = g(y) > 0$ have a continuous first derivative on $[c, d]$. Show that the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi\, g(y)\sqrt{1 + (g'(y))^2}\, dy.$$

In Exercises 57–64, find the area of the surface generated by revolving the curve about the indicated axis.

57. $x = \sqrt{y},\quad 0 \le y \le 2;\quad y$-axis
58. $x = y^3/3,\quad 0 \le y \le 1;\quad y$-axis
59. $x = y^{1/2} - (1/3)^{3/2},\quad 1 \le y \le 3;\quad y$-axis
60. $x = \sqrt{2y - 1},\quad (5/8) \le y \le 1;\quad y$-axis
61. $y = x^2,\quad 0 \le x \le 2;\quad x$-axis
62. $y = 3x - x^2,\quad 0 \le x \le 3;\quad x$-axis
63. $y = \sqrt{2x - x^2},\quad 0.5 \le x \le 1.5;\quad x$-axis
64. $y = \sqrt{x + 1},\quad 1 \le x \le 5;\quad x$-axis

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

65. **True or False** The volume of a solid of a known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is $\int_a^b A(x)\,dx$. Justify your answer.

66. **True or False** If the region enclosed by the y-axis, the line $y = 2$, and the curve $y = \sqrt{x}$ is revolved about the y-axis, the volume of the solid is given by the definite integral $\int_0^2 \pi y^2\,dy$. Justify your answer.

67. **Multiple Choice** The base of a solid S is the region enclosed by the graph of $y = \ln x$, the line $x = e$, and the x-axis. If the cross sections of S perpendicular to the x-axis are squares, which of the following gives the best approximation of the volume of S?
(A) 0.718 (B) 1.718 (C) 2.718
(D) 3.171 (E) 7.388

68. **Multiple Choice** Let R be the region in the first quadrant bounded by the graph of $y = 8 - x^{3/2}$, the x-axis, and the y-axis. Which of the following gives the best approximation of the volume of the solid generated when R is revolved about the x-axis?
(A) 60.3 (B) 115.2 (C) 225.4
(D) 319.7 (E) 361.9

69. **Multiple Choice** Let R be the region enclosed by the graph of $y = x^2$, the line $x = 4$, and the x-axis. Which of the following gives the best approximation of the volume of the solid generated when R is revolved about the y-axis?
(A) 64π (B) 128π (C) 256π
(D) 360 (E) 512

70. Multiple Choice Let R be the region enclosed by the graphs of $y = e^{-x}$, $y = e^{x}$, and $x = 1$. Which of the following gives the volume of the solid generated when R is revolved about the x-axis?

(A) $\int_0^1 (e^x - e^{-x})\,dx$ (B) $\int_0^1 (e^{2x} - e^{-2x})\,dx$

(C) $\int_0^1 (e^x - e^{-x})^2\,dx$ (D) $\pi \int_0^1 (e^{2x} - e^{-2x})\,dx$

(E) $\pi \int_0^1 (e^x - e^{-x})^2\,dx$

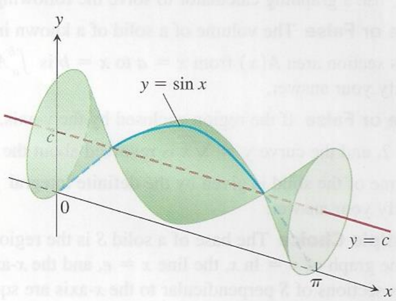
Explorations

71. *Max-Min* The arch $y = \sin x$, $0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in the figure.

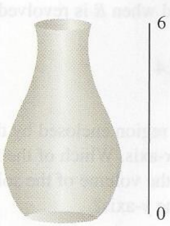
(a) Find the value of c that minimizes the volume of the solid. What is the minimum volume?

(b) What value of c in $[0, 1]$ maximizes the volume of the solid?

(c) **Writing to Learn** Graph the solid's volume as a function of c, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.



72. *A Vase* We wish to estimate the volume of a flower vase using only a calculator, a string, and a ruler. We measure the height of the vase to be 6 inches. We then use the string and the ruler to find circumferences of the vase (in inches) at half-inch intervals. (We list them from the top down to correspond with the picture of the vase.)



Circumferences	
5.4	10.8
4.5	11.6
4.4	11.6
5.1	10.8
6.3	9.0
7.8	6.3
9.4	

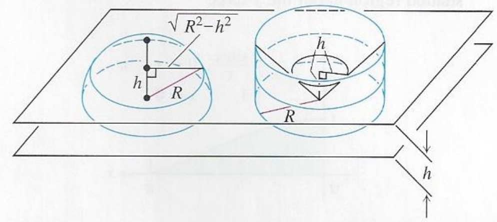
(a) Find the areas of the cross sections that correspond to the given circumferences.

(b) Express the volume of the vase as an integral with respect to y over the interval $[0, 6]$.

(c) Approximate the integral using the Trapezoidal Rule with $n = 12$.

Extending the Ideas

73. *Volume of a Hemisphere* Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross sections with the cross sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed, as suggested by the figure.



74. *Volume of a Torus* The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ $(b > a)$ to generate a solid shaped like a doughnut, called a *torus.* Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\,dy = \pi a^2/2$, since it is the area of a semicircle of radius a.)

75. *Filling a Bowl*

(a) ***Volume*** A hemispherical bowl of radius a contains water to a depth h. Find the volume of water in the bowl.

(b) ***Related Rates*** Water runs into a sunken concrete hemispherical bowl of radius 5 m at a rate of 0.2 m^3/sec. How fast is the water level in the bowl rising when the water is 4 m deep?

76. *Consistency of Volume Definitions* The volume formulas in calculus are consistent with the standard formulas from geometry in the sense that they agree on objects to which both apply.

(a) As a case in point, show that if you revolve the region enclosed by the semicircle $y = \sqrt{a^2 - x^2}$ and the x-axis about the x-axis to generate a solid sphere, the calculus formula for volume at the beginning of the section will give $(4/3)\pi a^3$ for the volume, just as it should.

(b) Use calculus to find the volume of a right circular cone of height h and base radius r.