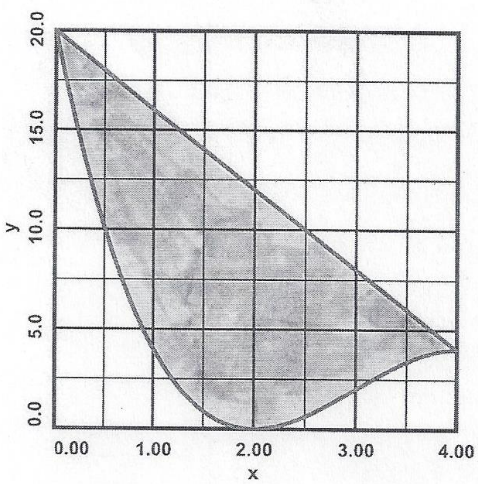
AP CALCULUS AB

AREAS BETWEEN CURVES

1) Find the shaded area between the curves

$f(x) = -4x + 20$ and

$g(x) = -x^3 + 9x^2 - 24x + 20$.



$$A = \int_0^4 [f(x) - g(x)]\,dx =$$

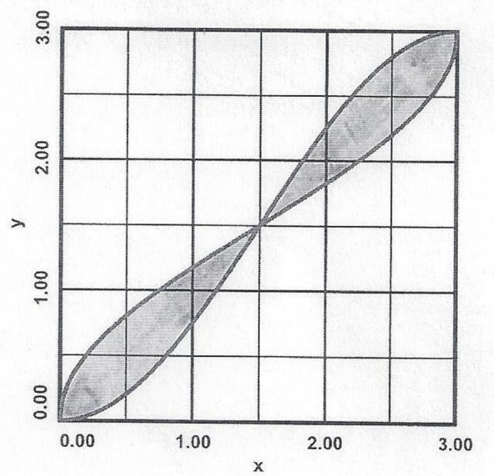
$$= \int_0^4 (x^3 - 9x^2 + 20x)\,dx =$$

$$= \left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_0^4 = 32$$

2) Find the shaded area between the curves

$$f(x) = 3 - \frac{3}{\pi}\cos^{-1}\left[\frac{1}{3}(2x-3)\right] \text{ and}$$

$$g(x) = \frac{3}{2}\left[1 - \cos\left(\frac{\pi x}{3}\right)\right].$$



Recall that

$$\int \cos^{-1}u \; du = u\cos^{-1}u - \sqrt{1-u^2}.$$

$$\int \cos^{-1}\left[\tfrac{1}{3}(2x-3)\right]dx = \int \cos^{-1}u \cdot \tfrac{3}{2}\,du = \tfrac{3}{2}\int \cos^{-1}u\,du$$

$$\boxed{u = \tfrac{1}{3}(2x-3),\quad \tfrac{du}{dx} = \tfrac{2}{3},\quad dx = \tfrac{3}{2}du}$$

$$= \tfrac{3}{2}u\cos^{-1}u - \tfrac{3}{2}\sqrt{1-u^2} =$$

$$= \tfrac{1}{2}(2x-3)\cos^{-1}\left[\tfrac{1}{3}(2x-3)\right] - \sqrt{3x-x^2}, \quad \int f(x)\,dx = 3x - \tfrac{3}{2\pi}(2x-3)\cos^{-1}\left[\tfrac{1}{3}(2x-3)\right] + \tfrac{3}{\pi}\sqrt{3x-x^2}$$

$$\int \cos\left(\tfrac{\pi x}{3}\right)dx = \int \cos u \cdot \tfrac{3}{\pi}\,du = \tfrac{3}{\pi}\int \cos u\,du = \tfrac{3}{\pi}\sin u = \tfrac{3}{\pi}\sin\left(\tfrac{\pi x}{3}\right),$$

$$\boxed{u = \tfrac{\pi x}{3},\quad \tfrac{du}{dx} = \tfrac{\pi}{3},\quad dx = \tfrac{3}{\pi}du} \qquad \int g(x)\,dx = \tfrac{3}{2}x - \tfrac{9}{2\pi}\sin\left(\tfrac{\pi x}{3}\right)$$

$$\int [f(x) - g(x)]\,dx = \tfrac{3}{2}x - \tfrac{3}{2\pi}(2x-3)\cos^{-1}\left[\tfrac{1}{3}(2x-3)\right] + \tfrac{3}{\pi}\sqrt{3x-x^2} + \tfrac{9}{2\pi}\sin\left(\tfrac{\pi x}{3}\right) = F(x)$$

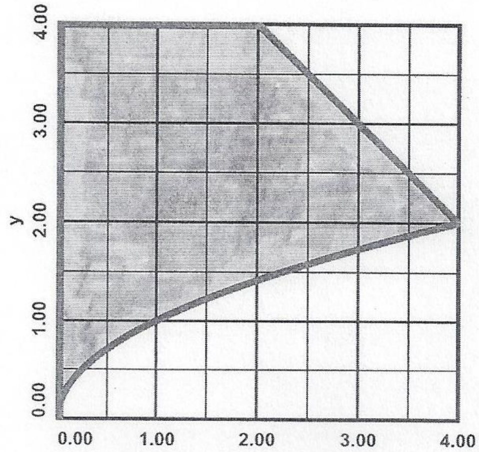
$$A = 2\int_0^{3/2}[f(x) - g(x)]\,dx = 2\left[F(x)\right]_0^{3/2} = 2\left[\tfrac{9}{\pi} - \tfrac{9}{4}\right] = 18\left[\tfrac{1}{\pi} - \tfrac{1}{4}\right] = 1.229\,577\,951$$

3) Find the shaded area two ways, *viz.*,

a) by integrating with respect to x, and

b) by integrating with respect to y.

The curve in the figure has the equation $f(x) = \sqrt{x}$.



$y = -x + b,\quad 2 = -4 + b,\quad b = 6$

$y = -x + 6$ is the equation of the diagonal line

(a) $A = \int_0^2 [4 - \sqrt{x}] + \int_2^4 [-x + 6 - \sqrt{x}]\,dx$

$= \int_0^2 4\,dx + \int_2^4 6\,dx - \int_0^2 \sqrt{x}\,dx - \int_2^4 \sqrt{x}\,dx$

$- \int_2^4 x\,dx = 8 + 12 - \int_0^4 \sqrt{x}\,dx - \int_2^4 x\,dx =$

$= 20 - \left[\frac{2}{3}\sqrt{x^3}\right]_0^4 - \left[\frac{x^2}{2}\right]_2^4 =$

$= 20 - \frac{16}{3} - 6 = 8\frac{2}{3}$

(b) $y = -x + 6,\quad x = -y + 6$

$y = \sqrt{x},\quad x = y^2$

$A = \int_0^2 y^2\,dy + \int_2^4 (-y + 6)\,dy = \left[\frac{y^3}{3}\right]_0^2 + \left[-\frac{y^2}{2} + 6y\right]_2^4 = 2\frac{2}{3} + 6 = 8\frac{2}{3}$