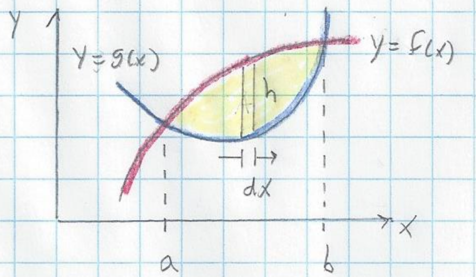
8.2. Areas Between Curves



$dA \equiv$ differential of area

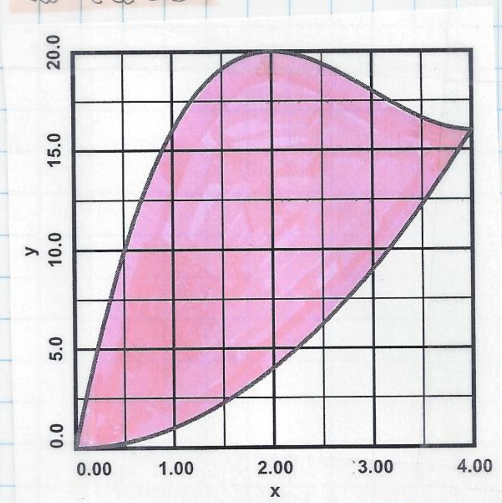
$dA =$ length $\cdot$ width $= h \cdot dx =$

$= [f(x) - g(x)] \cdot dx.$

So, the shaded area A is

$$\boxed{A = \int dA = \int_a^b [f(x) - g(x)]\, dx}$$

Example #1. Calculate the shaded area between $f(x) = x^3 - 9x^2 + 24x$ and $g(x) = x^2$.

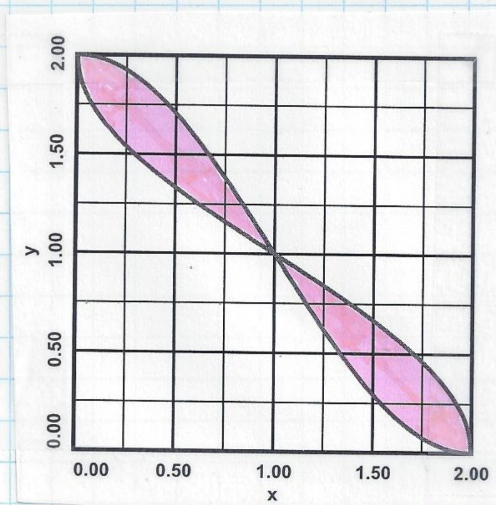


SOLUTION: $f(x) - g(x) = x^3 - 10x^2 + 24x$

$$A = \int_0^4 (x^3 - 10x^2 + 24x)\, dx =$$

$$= \left[\frac{x^4}{4} - \frac{10}{3}x^3 + 12x^2\right]_0^4 = 42\tfrac{2}{3}$$

Example #2. Calculate the shaded area between $f(x) = \cos\left(\frac{\pi x}{2}\right) + 1$ and $g(x) = \frac{2}{\pi}\cos^{-1}(x-1)$.



SOLUTION:

$\int \cos\left(\frac{\pi x}{2}\right) dx$: $u = \frac{\pi x}{2}$, $du = \frac{\pi}{2} dx$,

$dx = \frac{2}{\pi} du$, $\int \cos\left(\frac{\pi x}{2}\right) dx =$

$= \int \cos u \cdot \frac{2}{\pi} du = \frac{2}{\pi}\int \cos u\, du =$

$= \frac{2}{\pi}\sin u = \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)$

$\int \cos^{-1}(x-1)\, dx$; $u = x - 1$, $du = dx$

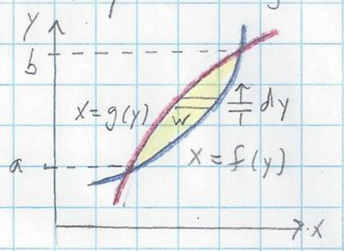
$\int \cos^{-1}(x-1)\, dx = \int \cos^{-1} u\, du$. Recall that $\int \cos^{-1} u\, du = u\cos^{-1} u - \sqrt{1-u^2} \Rightarrow$

$\int \cos^{-1}(x-1)\,dx = (x-1)\cos^{-1}(x-1) - \sqrt{2x-x^2}$, $A = 2\int_0^1 [f(x)-g(x)]\,dx =$

$= 2\int_0^1 f(x)\,dx - 2\int_0^1 g(x)\,dx = \left[\frac{4}{\pi}\sin\left(\frac{\pi x}{2}\right) + 2x - \frac{4}{\pi}(x-1)\cos^{-1}(x-1) + \frac{4}{\pi}\sqrt{2x-x^2}\right]_0^1$

$= \frac{8}{\pi} - 2 = 0.546\ 479\ 0895$

One may also integrate with respect to y ...



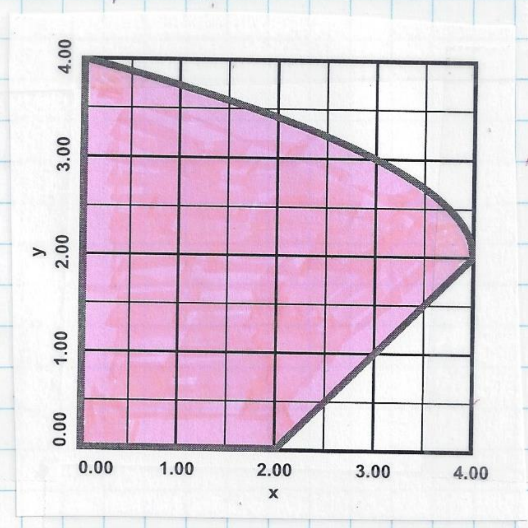
$dA = w\,dy = [f(y) - g(y)]\,dy \Rightarrow$ the shaded area A is

$$\boxed{A = \int dA = \int_a^b [f(y) - g(y)]\,dA}$$

Example #3. Calculate the shaded area two ways, viz.,

(a) by integrating with respect to x, and (b) by integrating with respect to y

the curve in the figure has the equation $y = 2 + \sqrt{4-x}$.



SOLUTION:

The diagonal line in the figure has the equation $y = x - 2$.

(a) We need to break the integral into two parts...

$f(x) = 2 + \sqrt{4-x}$, $g(x) = x - 2$,

$A = \int_0^2 f(x)\,dx + \int_2^4 [f(x) - g(x)]\,dx =$

$= \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx - \int_2^4 g(x)\,dx =$

$= \int_0^4 f(x)\,dx - \int_2^4 g(x)\,dx$. $\int \sqrt{4-x}\,dx$: $u = 4-x$, $du = -dx$, $dx = -du$

$\int \sqrt{4-x}\,dx = \int \sqrt{u} \cdot -du = -\int \sqrt{u}\,du = -\frac{2}{3}\sqrt{u^3} = -\frac{2}{3}\sqrt{(4-x)^3}$

$$A = \left[2x - \frac{2}{3}\sqrt{(4-x)^3}\right]_0^4 - \left[\frac{1}{2}x^2 - 2x\right]_2^4 = \left[13\tfrac{1}{3}\right] - \left[2\right] = 11\tfrac{1}{3}$$

(b) Invert the functions... $y = x-2$, $x = y+2 \Rightarrow f(y) = y+2$

$y = 2 + \sqrt{4-x}$, $\sqrt{4-x} = y-2$, $4-x = (y-2)^2 = y^2-4y+4$, $x-4 = -y^2+4y-4$,

$x = -y^2+4y \Rightarrow g(y) = -y^2+4y$. Again, we need to break the integral into two parts...

$$A = \int_0^2 f(y)\,dy + \int_2^4 g(y)\,dy = \int_0^2 (y+2)\,dy + \int_2^4 (-y^2+4y)\,dy =$$

$$= \left[\frac{y^2}{2} + 2y\right]_0^2 + \left[-\frac{y^3}{3} + 2y^2\right]_2^4 = \left[6\right] + \left[5\tfrac{1}{3}\right] = 11\tfrac{1}{3}$$