1) Verify the integral

$$\int \cos^3\theta \, d\theta = \sin\theta - \frac{1}{3}\sin^3\theta$$

by differentiation.

Hint: use $\sin^2\theta + \cos^2\theta = 1$.

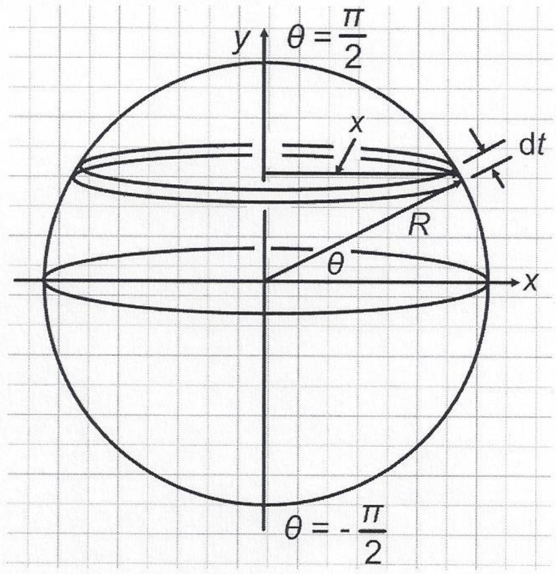
$$\frac{d}{d\theta}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right] =$$

$$= \cos\theta - \frac{1}{\cancel{3}}\cdot\cancel{3}\sin^2\theta\cdot\cos\theta =$$

$$= \cos\theta\,(1-\sin^2\theta) =$$

$$= \cos\theta\,(\cos^2\theta) = \cos^3\theta$$

Surface Area of a Sphere:



The differential of surface area dS is the ring shown on the surface of the sphere. On the differential scale, everything is linear, *i.e.,* flat. Accordingly

$$dS = \text{length}\cdot\text{width} = \text{circumference}\cdot dt = $$
$$= 2\pi x \cdot R d\theta\,.$$

2) Use $x = R\cos\theta$ to set up an integral for $S = \int dS$ in terms of R and θ, and then integrate from $\theta = -{}^{\pi}/_{2}$ to $\theta = {}^{\pi}/_{2}$ to find the well-known formula for the surface area of a sphere.

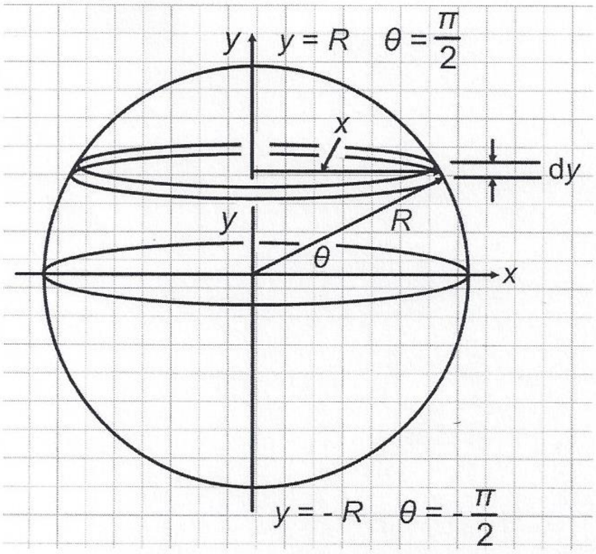
$$dS = 2\pi R\cos\theta\cdot R\,d\theta = 2\pi R^2\cos\theta\,d\theta$$

$$S = \int dS = 2\pi R^2\int_{-\frac{\pi}{2}}^{\pi/2}\cos\theta\,d\theta =$$

$$= 2\pi R^2\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = 2\pi R^2\cdot 2$$

$$= 4\pi R^2$$

Volume of a Sphere:



The differential of volume $\mathrm{d}V$ is the cylinder in the sphere as shown on the figure. So,

$\mathrm{d}V = \text{base area} \cdot \text{height} = \pi x^2 \cdot \mathrm{d}y$.

3) Use $x = R\cos\theta$ and $y = R\sin\theta$ to set up an integral for $V = \int \mathrm{d}V$ in terms of R and θ, and then integrate from $\theta = -\pi/2$ to $\theta = \pi/2$ to find the well-known formula for the volume of a sphere.

Hint: you will need to use the integral from problem **1**.

$Y = R\sin\theta, \quad dy = R\cos\theta\, d\theta,$

$dV = \pi R^2 \cos^2\theta \cdot R\cos\theta\, d\theta =$

$= \pi R^3 \cos^3\theta\, d\theta$

$V = \int dV = \pi R^3 \int_{-\pi/2}^{\pi/2} \cos^3\theta\, d\theta =$

$= \pi R^3 \left[\sin\theta - \frac{1}{3}\sin^3\theta \right]_{\pi/2}^{-\pi/2} =$

$= \pi R^3 \cdot \frac{4}{3} = \frac{4}{3}\pi R^3$