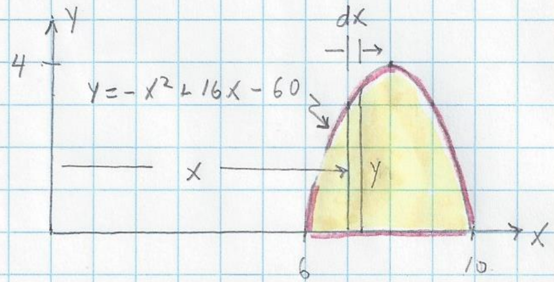
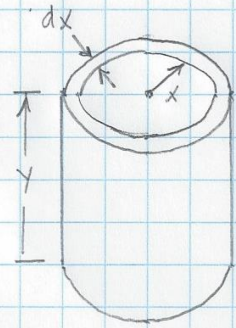
8.3. Cylindrical Shells

Example #1. Find the volume of the solid generated by rotating the shaded area around the y-axis.





SOLUTION:

The differential of volume dV is the cylindrical shell pictured.

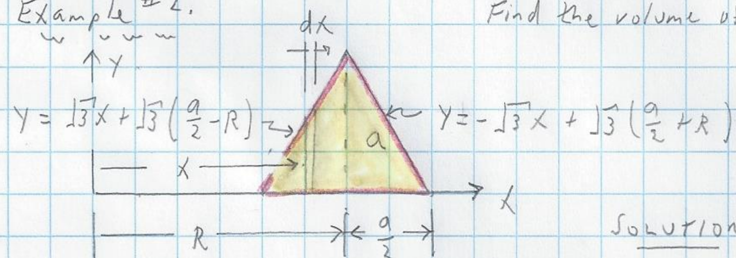
Again, on the differential scale, things are straight and flat $\Rightarrow$

$$dV=(\text{base area})\cdot(\text{height})=(\text{length})\cdot(\text{width})\cdot(\text{height})=(2\pi x)(dx)(y)=$$

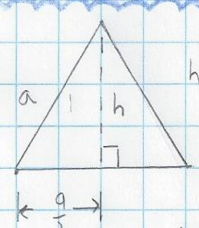
$$=2\pi x\,dx\,y=2\pi x\,dx\,(-x^2+16x-60)=2\pi(-x^3+16x^2-60x)\,dx,$$

$$V=\int dV=2\pi\int_6^{10}(-x^3+16x^2-60x)\,dx=2\pi\left[-\tfrac{1}{4}x^4+\tfrac{16}{3}x^3-30x^2\right]_6^{10}=170\tfrac{2}{3}\pi$$

Example #2. Find the volume of the solid generated by rotating the equilateral triangle of side length a around the y-axis.



SIDE NOTE:



$h=\frac{\sqrt{3}}{2}a$

The area of the triangle is

$$\frac{1}{2}(\text{base})(\text{height})=$$

$$=\frac{1}{2}\cdot a\cdot\frac{\sqrt{3}}{2}a=\frac{\sqrt{3}}{4}a^2$$

SOLUTION:

$$dV=2\pi x\,dx\,y$$

$R-\frac{a}{2}\le x\le R$:

$$dV=2\pi x\left[\sqrt{3}x+\sqrt{3}\left(\frac{a}{2}-R\right)\right]dx$$

$R\le x\le R+\frac{a}{2}$:

$$dV=2\pi x\left[-\sqrt{3}x+\sqrt{3}\left(\frac{a}{2}+R\right)\right]dx$$

$$V = 2\sqrt{3}\,\pi \int_{R-\frac{a}{2}}^{R} \left[x^2 + \left(\tfrac{a}{2} - R\right)x\right] dx + 2\sqrt{3}\,\pi \int_{R}^{R+\frac{a}{2}} \left[-x^2 + \left(\tfrac{a}{2} + R\right)x\right] dx =$$

$$= \frac{\sqrt{3}}{3}\pi \left[2x^3 + 3\left(\tfrac{a}{2} - R\right)x^2\right]_{R-\frac{a}{2}}^{R} + \frac{\sqrt{3}}{3}\pi \left[-2x^3 + 3\left(\tfrac{a}{2} + R\right)x^2\right]_{R}^{R+\frac{a}{2}} =$$

$$= \frac{\sqrt{3}}{3}\pi \left[2R^3 - 3\left(R - \tfrac{a}{2}\right)R^2 + \left(R - \tfrac{a}{2}\right)^3\right]$$

$$+ \frac{\sqrt{3}}{3}\pi \left[2R^3 - 3\left(R + \tfrac{a}{2}\right)R^2 + \left(R + \tfrac{a}{2}\right)^3\right] =$$

$$= \frac{\sqrt{3}}{3}\pi \left[-2R^3 + \left(R - \tfrac{a}{2}\right)^3 + \left(R + \tfrac{a}{2}\right)^3\right] = \frac{\sqrt{3}}{3}\pi \left[\tfrac{3}{2}Ra^2\right] = \frac{\sqrt{3}}{2}\pi R a^2$$

Note: The result may be written as $2\pi R \left[\frac{\sqrt{3}}{4}a^2\right] =$

$= 2\pi R \cdot$ (area of the triangle), where R is the rotation radius running to the center of the triangle. This result is true in general. This is the Pappus - Guldinus Theorem.