THE BOUNDARY ELEMENT METHOD FOR TWO-DIMENSIONAL LINEAR ELASTICITY

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0. Contents

1. Hooke's Law

Hooke's Law is

$$\varepsilon_{xx} = S\left[\,(1-\nu^*)\sigma_{xx} - \nu^*\sigma_{yy}\,\right] , \qquad \varepsilon_{yy} = S\left[\,(1-\nu^*)\sigma_{yy} - \nu^*\sigma_{xx}\,\right] , \qquad \varepsilon_{xy} = S\sigma_{xy} \ , \tag{1}$$

where, respectively, ε_{ij} and σ_{ij} are the components of the strain and stress tensors, and

$$S = \frac{1+\nu}{E} \ , \quad \nu^* = \nu \ \text{(plane strain)} \ , \quad \nu^* = \frac{\nu}{1+\nu} \ \text{(plane stress).} \tag{2}$$

In eqns. (2), E is Young's modulus, and ν is Poisson's ratio. Also,

$$\begin{aligned} &\text{plane strain:} \quad \varepsilon_{zz} = 0 \ , \quad \sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right) \\ &\text{plane stress:} \quad \sigma_{zz} = 0 \ , \quad \varepsilon_{zz} = -\frac{\nu}{1-\nu}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) \end{aligned} \ . \tag{3}$$

Inversely, eqns. (1) are

$$\begin{aligned} \sigma_{xx} &= \frac{1}{S(1-2\nu^*)}\left[\,(1-\nu^*)\varepsilon_{xx} + \nu^*\varepsilon_{yy}\,\right] , \\ \sigma_{yy} &= \frac{1}{S(1-2\nu^*)}\left[\,(1-\nu^*)\varepsilon_{yy} + \nu^*\varepsilon_{xx}\,\right] , \\ \sigma_{xy} &= \frac{1}{S}\,\varepsilon_{xy} \ . \end{aligned} \tag{4}$$

Finally, eqns. (4) may be written in tensorial form as

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} \ , \qquad L_{ijkl} = \frac{1}{S}\left[I_{ijkl} + \frac{\nu^*}{1-2\nu^*}\delta_{ij}\delta_{kl}\right] , \qquad I_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right) \ , \tag{5}$$

where δ_{ij} is the two-dimensional identity matrix (or Kronecker delta).

2. Unit Point Load in an Infinite Plate in the x-Direction

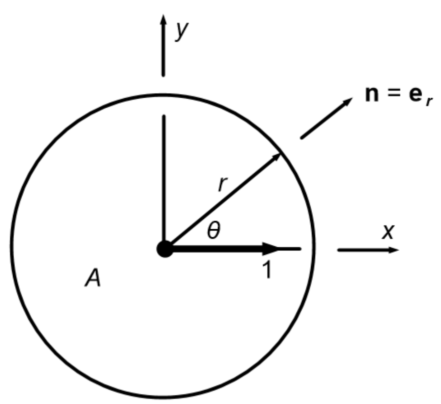

Figure 1. Unit point load applied at the origin of an infinite plate in the x-direction.

Figure 1 at left depicts a unit point load applied at the origin of an infinite plate in the x-direction. In polar coordinates, it is seen that the stresses

$$\begin{aligned} \sigma_{rr} &= -\frac{k_1}{r}\cos\theta \ , \\ \sigma_{r\theta} &= \frac{k_2}{r}\sin\theta \ , \\ \sigma_{\theta\theta} &= \frac{k_2}{r}\cos\theta \end{aligned} \tag{6}$$

are of the form which may be in equilibrium with the point load, with $k_1 > 0$ and $k_2 > 0$. In the figure, $\mathbf{n}$ is the outward-pointing unit normal on the circle of radius r centered at the origin, which is the same as the radial unit base vector $\boldsymbol{e}_r$. Note that the stress components (6) satisfy the equilibrium equations

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\theta,\theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \ , \qquad \sigma_{r\theta,r} + \frac{1}{r}\sigma_{\theta\theta,\theta} + \frac{2}{r}\sigma_{r\theta} = 0 \tag{7}$$

identically. The traction vector on the circle of Fig. 1 is $\mathbf{T} = T_r\mathbf{e}_r + T_\theta\mathbf{e}_\theta$, with $T_r = \sigma_{rr}$ and $T_\theta = \sigma_{r\theta}$. Or, in rectangular components via

$$\begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T_r \\ T_\theta \end{bmatrix} \tag{8}$$

the traction components are

$$T_x = -\frac{1}{r}(k_1\cos^2\theta + k_2\sin^2\theta) \ , \qquad T_y = -\frac{1}{r}(k_1 - k_2)\sin\theta\cos\theta \ . \tag{9}$$

Equilibrium requires that

$$\int_0^{2\pi} T_x r\mathrm{d}\theta = -1 \ , \qquad \int_0^{2\pi} T_y r\mathrm{d}\theta = 0 \ . \tag{10}$$

The first of eqns. (10) gives

$$k_1 + k_2 = \frac{1}{\pi} \tag{11}$$

while the second of eqns. (10) is satisfied identically.

Using Hooke's Law (1), the stresses (6) give the strains

$$\varepsilon_{rr} = -S\,[\,(1-\nu^*)k_1 + \nu^*k_2\,]\,\frac{\cos\theta}{r} \ , \qquad \varepsilon_{\theta\theta} = S\,[\,\nu^*k_1 + (1-\nu^*)k_2\,]\,\frac{\cos\theta}{r} \ ,$$
$$\varepsilon_{r\theta} = Sk_2\frac{\sin\theta}{r} \ . \tag{12}$$

In polar coordinates the strain-displacement relations are

$$\varepsilon_{rr} = u_{r,r} \ , \qquad \varepsilon_{\theta\theta} = \frac{1}{r}u_{\theta,\theta} + \frac{1}{r}u_r \ , \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}u_{r,\theta} + u_{\theta,r} - \frac{1}{r}u_\theta\right) \ , \tag{13}$$

where u_r and u_θ are the displacement components. Now, using the first of eqns. (12) to integrate the first of eqns. (13) yields

$$u_r = -S\,[\,(1-\nu^*)k_1 + \nu^*k_2\,]\,\ln r\cos\theta + f'(\theta) \ , \tag{14}$$

where $f(\theta)$ is a function of integration. Next, using the second of eqns. (12) and eqn. (14) to integrate the second of eqns. (13), one obtains

$$u_\theta = S[\,\nu^*k_1 + (1-\nu^*)k_2\,]\,\sin\theta + S[\,(1-\nu^*)k_1 + \nu^*k_2\,]\,\ln r\sin\theta - f(\theta) + g(r) \ , \tag{15}$$

where $g(r)$ is another function of integration. Finally, putting the third of eqns. (12) and eqns. (14) and (15) into the third of eqns. (13), one sees that the functions of integration may be chosen as $f(\theta) = 0$ and $g(r) = 0$ if

$$(1-2\nu^*)k_1 - (3-2\nu^*)k_2 = 0 \,. \tag{16}$$

So, via eqns. (11) and (16), the constants are

$$k_1 = \frac{(3-2\nu^*)}{4\pi(1-\nu^*)} \ , \qquad k_2 = \frac{(1-2\nu^*)}{4\pi(1-\nu^*)} \ . \tag{17}$$

With eqns. (17), the displacements (14) and (15) become

$$u_r = -\frac{(3-4\nu^*)S}{4\pi(1-\nu^*)} \ln r \cos\theta \ , \qquad u_\theta = \frac{S}{4\pi(1-\nu^*)}[\,1+(3-4\nu^*)\ln r\,]\ \sin\theta \ . \qquad (18)$$

One notes that, with eqns. (17), the stresses (6) agree with the expressions for plane stress given on pg. 129 of the text Theory of Elasticity, 3rd ed., S.P. Timoshenko and J.N. Goodier, McGraw-Hill (1970).

Transforming eqns. (18) to rectangular coordinates gives, *cf.,* eqns. (8),

$$u_x = -\frac{S}{4\pi(1-\nu^*)}\left[(3-4\nu^*)\ln r + \frac{y^2}{r^2}\right] , \qquad u_y = \frac{S}{4\pi(1-\nu^*)}\left[\frac{xy}{r^2}\right] , \qquad (19)$$

where $x = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{x^2+y^2}$. Noting that $r_{,x} = x/r$ and $r_{,y} = y/r$, using eqns. (19) and the strain-displacement relations $\varepsilon_{ij} = \left(u_{i,j}+u_{j,i}\right)/2$, one obtains the strains

$$\begin{aligned}
\varepsilon_{xx} &= -\frac{S}{4\pi(1-\nu^*)}\left[(3-4\nu^*)\frac{x}{r^2} - 2\frac{xy^2}{r^4}\right] , \\
\varepsilon_{yy} &= \frac{S}{4\pi(1-\nu^*)}\left[\frac{x}{r^2} - 2\frac{xy^2}{r^4}\right] , \\
\varepsilon_{xy} &= -\frac{S}{4\pi(1-\nu^*)}\left[2(1-\nu^*)\frac{y}{r^2} + \frac{x^2y}{r^4} - \frac{y^3}{r^4}\right] ,
\end{aligned} \qquad (20)$$

and then via eqns. (4) and (20), the stresses are

$$\begin{aligned}
\sigma_{xx} &= -\frac{1}{4\pi(1-\nu^*)}\left[(3-2\nu^*)\frac{x}{r^2} - 2\frac{xy^2}{r^4}\right] , \\
\sigma_{yy} &= \frac{1}{4\pi(1-\nu^*)}\left[(1-2\nu^*)\frac{x}{r^2} - 2\frac{xy^2}{r^4}\right] , \\
\sigma_{xy} &= -\frac{1}{4\pi(1-\nu^*)}\left[2(1-\nu^*)\frac{y}{r^2} + \frac{x^2y}{r^4} - \frac{y^3}{r^4}\right] .
\end{aligned} \qquad (21)$$

3. Unit Point Load in an Infinite Plate in the *y*-Direction

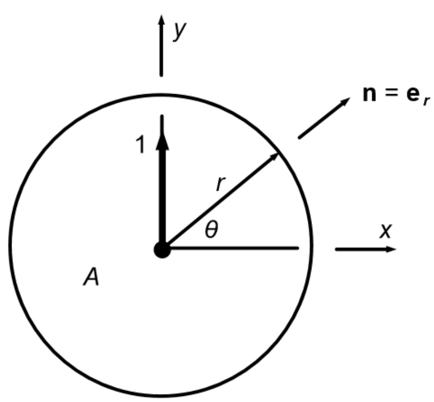

Figure 2. Unit point load applied at the origin of an infinite plate in the y-direction.

In Fig. 2 at left, the unit point load is now applied in the y-direction. By analogy with eqns. (6), one sees that the stress components

$$\begin{aligned}
\sigma_{rr} &= -\frac{k_1}{r}\sin\theta \ , \\
\sigma_{r\theta} &= -\frac{k_2}{r}\cos\theta \ , \\
\sigma_{\theta\theta} &= \frac{k_2}{r}\sin\theta
\end{aligned} \qquad (22)$$

can be in equilibrium with the point load, where once again, $k_1 > 0$ and $k_2 > 0$. As before, the stresses (22) satisfy the equilibrium eqns. (7) identically. Also, in rectangular components, the traction vector acting on the circle in Fig. 2 is

$$T_x = -\frac{1}{r}\,(k_1-k_2)\sin\theta\cos\theta \ , \qquad T_y = -\frac{1}{r}\,(k_1\sin^2\theta + k_2\cos^2\theta) \ . \qquad (23)$$

Equilibrium also requires that

$$\int_0^{2\pi} T_x r \mathrm{d}\theta = 0 \ , \qquad \int_0^{2\pi} T_y r \mathrm{d}\theta = -1 \ . \tag{24}$$

The first of eqns. (24) is satisfied identically, while the second of eqns. (24) gives the relation (11) above. Omitting the details, when one uses eqns. (22) to calculate the strains, and then when these strains are integrated to get the displacements, the relation (16) above is also obtained. In short, here the constants k_1 and k_2 are the same as in eqn. (17) above.

Omitting the details then, one obtains

$$u_x = \frac{S}{4\pi(1-\nu^*)}\left[\frac{xy}{r^2}\right] , \qquad u_y = -\frac{S}{4\pi(1-\nu^*)}\left[(3-4\nu^*)\ln r + \frac{x^2}{r^2}\right] , \tag{25}$$

and

$$\begin{aligned}
\sigma_{xx} &= \frac{1}{4\pi(1-\nu^*)}\left[(1-2\nu^*)\frac{y}{r^2} - 2\frac{x^2 y}{r^4}\right] , \\
\sigma_{yy} &= -\frac{1}{4\pi(1-\nu^*)}\left[(3-2\nu^*)\frac{y}{r^2} - 2\frac{x^2 y}{r^4}\right] , \\
\sigma_{xy} &= -\frac{1}{4\pi(1-\nu^*)}\left[2(1-\nu^*)\frac{x}{r^2} + \frac{xy^2}{r^4} - \frac{x^3}{r^4}\right] .
\end{aligned} \tag{26}$$

4. The Green's Functions

Let g_j^i denote the j-direction displacement due to a unit point load applied in the i-direction. Additionally, the unit point loads are taken to be applied away from the origin at the location (x_0, y_0). So, with the notation $X = x - x_0$, $Y = y - y_0$ and $r = \sqrt{X^2 + Y^2}$, eqns. (19) and (25) give

$$\begin{aligned}
g_x^x &= -\frac{S}{4\pi(1-\nu^*)}\left[(3-4\nu^*)\ln r + \frac{Y^2}{r^2}\right] , \qquad g_y^x = g_x^y = \frac{S}{4\pi(1-\nu^*)}\left[\frac{XY}{r^2}\right] , \\
g_y^y &= -\frac{S}{4\pi(1-\nu^*)}\left[(3-4\nu^*)\ln r + \frac{X^2}{r^2}\right] .
\end{aligned} \tag{27}$$

Similarly, let G_{ij}^k be the stress tensor due to a unit point load in the k-direction. Equations (21) and (26) then are

$$\begin{aligned}
G_{xx}^x &= -\frac{1}{4\pi(1-\nu^*)}\left[(3-2\nu^*)\frac{X}{r^2} - 2\frac{XY^2}{r^4}\right] , \qquad G_{yy}^x = \frac{1}{4\pi(1-\nu^*)}\left[(1-2\nu^*)\frac{X}{r^2} - 2\frac{XY^2}{r^4}\right] , \\
G_{xy}^x &= G_{yx}^x = -\frac{1}{4\pi(1-\nu^*)}\left[2(1-\nu^*)\frac{Y}{r^2} + \frac{X^2 Y}{r^4} - \frac{Y^3}{r^4}\right] , \\
G_{xx}^y &= \frac{1}{4\pi(1-\nu^*)}\left[(1-2\nu^*)\frac{Y}{r^2} - 2\frac{X^2 Y}{r^4}\right] , \qquad G_{yy}^y = -\frac{1}{4\pi(1-\nu^*)}\left[(3-2\nu^*)\frac{Y}{r^2} - 2\frac{X^2 Y}{r^4}\right] , \\
G_{xy}^y &= G_{yx}^y = -\frac{1}{4\pi(1-\nu^*)}\left[2(1-\nu^*)\frac{X}{r^2} + \frac{XY^2}{r^4} - \frac{X^3}{r^4}\right] .
\end{aligned} \tag{28}$$

Equations (27) and (28) are termed the Green's functions for two-dimensional linear elasticity.

Owing to eqns. (10) and (24) and the Divergence Theorem, the functions (28) possess the property

$$\int_A G_{ij,i}^k \mathrm{d}A\, \mathbf{e}_j = \oint_t n_i G_{ij}^k \mathrm{d}t\, \mathbf{e}_j = -c\mathbf{e}_k \ , \tag{29}$$

where A is the two-dimensional domain of interest, t is the boundary of A (with positive arc length being measured in the counterclockwise fashion), n_i is the outward pointing unit normal on t, and $\mathbf{e}_i$ are the Cartesian base vectors. The constant in eqn. (29) is $c = 1$ if the singularity $\mathbf{x}_0$ is in A, and $c = 0$ if $\mathbf{x}_0$ is outside of A. Also, since G_{ij}^k satisfies equilibrium, *i.e.,* $G_{ij,i}^k = 0$ everywhere except at $\mathbf{x}_0$ (where it is singular), then consequently, one may multiply eqn. (29) by a displacement field u_j underneath the integral sign to see

$$\int_A u_j G_{ij,i}^k \mathrm{d}A = -cu_k(\mathbf{x}_0)\ . \tag{30}$$

5. The Reciprocal Theorem

Let u_j be a (non-singular) displacement field with corresponding stresses σ_{ij} that satisfy equilibrium so that $\sigma_{ij,i} = 0$. Multiplication of this equation by the Green's function then gives $g_j^k \sigma_{ij,i} = 0$. Thus, by the product rule of differentiation

$$\left(g_j^k \sigma_{ij}\right)_{,i} = g_{j,i}^k \sigma_{ij} + g_j^k \sigma_{ij,i} = g_{j,i}^k \sigma_{ij} \ . \tag{31}$$

Now, since the elasticity tensor in eqn. (5) possesses the symmetries $L_{ijkl} = L_{klij} = L_{jikl} = L_{ijlk}$,

$$g_{j,i}^k \sigma_{ij} = g_{j,i}^k L_{ijpq} u_{q,p} = G_{pq}^k u_{q,p} = G_{ij}^k u_{j,i} \tag{32}$$

so that eqn. (31) becomes

$$\left(g_j^k \sigma_{ij}\right)_{,i} = G_{ij}^k u_{j,i} \ . \tag{33}$$

Again, via the product rule of differentiation, $\left(G_{ij}^k u_j\right)_{,i} = G_{ij,i}^k u_j + G_{ij}^k u_{j,i}$, which when put into eqn. (33) yields

$$-G_{ij,i}^k u_j = -\left(G_{ij}^k u_j\right)_{,i} + \left(g_j^k \sigma_{ij}\right)_{,i} \ . \tag{34}$$

Finally, integrating eqn. (34) over A, and then using the Divergence Theorem and eqn. (30), one obtains the Reciprocal Theorem, *viz.,*

$$cu_k(\mathbf{x}_0) = -\oint_t n_i G_{ij}^k u_j \mathrm{d}t + \oint_t g_j^k T_j \mathrm{d}t \ , \tag{35}$$

where $T_j = n_i \sigma_{ij}$ is the traction vector.

6. The Boundary Element Method – Displacement Solution

Figure 3 below shows a schematic of a boundary element. The element is a straight-line segment joining its two nodes 0 and 1, each of which has nodal displacements u_i^I and nodal tractions T_i^I, where I is the index of the node. In the element, the displacements and tractions are represented by the linear interpolations

$$u_i = f^I u_i^I \qquad T_i = f^I T_i^I$$
$$f^0 = \frac{L-t}{L} \qquad f^1 = \frac{t}{L} \quad . \tag{36}$$

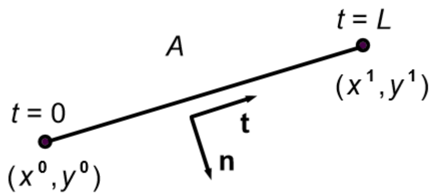

Figure 3. Schematic of a boundary element showing its unit tangent vector **t** and its outward pointing unit normal vector **n**.

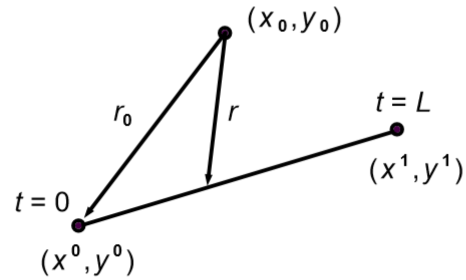

Figure 4. Geometry for evaluating the Green's functions along the length of the boundary element as a function of t.

When solving for the nodal boundary displacements and tractions, the singularity $\mathbf{x}_0$ will be taken as being outside of the domain A, so that $c = 0$ in eqn. (35). Putting the interpolations (36) into eqn. (35) then gives

$$0 = -A_{kjI} u_j^I + B_{kjI} T_j^I \ , \qquad A_{kjI} = \int_0^L n_i G_{ij}^k f^I \mathrm{d}t \ , \qquad B_{kjI} = \int_0^L g_j^k f^I \mathrm{d}t \ . \tag{37}$$

To evaluate the Green's functions in eqns. (37) as a function of s within an element, the geometry shown in Fig. 4 is used. So, along the element

$$\mathbf{r}_0 = (x^0 - x_0)\mathbf{e}_x + (y^0 - y_0)\mathbf{e}_y = r_{0x}\mathbf{e}_x + r_{0y}\mathbf{e}_y \ ,$$
$$x - x_0 = X = t_x t + r_{0x} \ , \qquad y - y_0 = Y = t_y t + r_{0y} \ . \tag{38}$$

Also, by using $r^2 = X^2 + Y^2$,

$$Q \equiv r^2 = t^2 + 2\beta t + r_0^2 \ , \qquad \beta = t_x r_{0x} + t_y r_{0y} \ , \qquad r_0^2 = r_{0x}^2 + r_{0y}^2 \ . \tag{39}$$

Now, substitution of eqns. (38) and (39) into eqns. (28), *i.e.,* the formulas for G_{ij}^k, gives the expressions

$$\begin{aligned}
G_{xx}^x &= a_1 \frac{t}{Q} + a_2 \frac{1}{Q} + a_3 \frac{t^3}{Q^2} + a_4 \frac{t^2}{Q^2} + a_5 \frac{t}{Q^2} + a_6 \frac{1}{Q^2} \ , \\
G_{xy}^x = G_{yx}^x &= a_7 \frac{t}{Q} + a_8 \frac{1}{Q} + a_9 \frac{t^3}{Q^2} + a_{10} \frac{t^2}{Q^2} + a_{11} \frac{t}{Q^2} + a_{12} \frac{1}{Q^2} \ , \\
G_{yy}^x &= a_{13} \frac{t}{Q} + a_{14} \frac{1}{Q} - a_3 \frac{t^3}{Q^2} - a_4 \frac{t^2}{Q^2} - a_5 \frac{t}{Q^2} - a_6 \frac{1}{Q^2} \ , \\
G_{xx}^y &= a_{15} \frac{t}{Q} + a_{16} \frac{1}{Q} + a_{17} \frac{t^3}{Q^2} + a_{18} \frac{t^2}{Q^2} + a_{19} \frac{t}{Q^2} + a_{20} \frac{1}{Q^2} \ , \\
G_{xy}^y = G_{yx}^y &= a_{21} \frac{t}{Q} + a_{22} \frac{1}{Q} + a_{23} \frac{t^3}{Q^2} + a_{24} \frac{t^2}{Q^2} + a_{25} \frac{t}{Q^2} + a_{26} \frac{1}{Q^2} \ , \\
G_{yy}^y &= a_{27} \frac{t}{Q} + a_{28} \frac{1}{Q} - a_{17} \frac{t^3}{Q^2} - a_{18} \frac{t^2}{Q^2} - a_{19} \frac{t}{Q^2} - a_{20} \frac{1}{Q^2} \ ,
\end{aligned} \tag{40}$$

where the values of the constants a_1 through a_{28} are listed below in Sec. 9. Next, performing the dot product $n_i G_{ij}^k$ gives, where the values of the constants b_1 through b_{24} are listed below in Sec. 10,

$$
\begin{aligned}
n_i G_{ix}^x &= b_1 \frac{t}{Q} + b_2 \frac{1}{Q} + b_3 \frac{t^3}{Q^2} + b_4 \frac{t^2}{Q^2} + b_5 \frac{t}{Q^2} + b_6 \frac{1}{Q^2} \ , \\
n_i G_{iy}^x &= b_7 \frac{t}{Q} + b_8 \frac{1}{Q} + b_9 \frac{t^3}{Q^2} + b_{10} \frac{t^2}{Q^2} + b_{11} \frac{t}{Q^2} + b_{12} \frac{1}{Q^2} \ , \\
n_i G_{ix}^y &= b_{13} \frac{t}{Q} + b_{14} \frac{1}{Q} + b_{15} \frac{t^3}{Q^2} + b_{16} \frac{t^2}{Q^2} + b_{17} \frac{t}{Q^2} + b_{18} \frac{1}{Q^2} \ , \\
n_i G_{iy}^y &= b_{19} \frac{t}{Q} + b_{20} \frac{1}{Q} + b_{21} \frac{t^3}{Q^2} + b_{22} \frac{t^2}{Q^2} + b_{23} \frac{t}{Q^2} + b_{24} \frac{1}{Q^2} \ ,
\end{aligned}
\tag{41}
$$

and then substituting eqns. (38) and (39) into eqns. (27) yields

$$
\begin{aligned}
g_x^x &= b_{25} \ln Q + b_{26} \frac{t^2}{Q} + b_{27} \frac{t}{Q} + b_{28} \frac{1}{Q} \ , \\
g_y^x = g_x^y &= b_{29} \frac{t^2}{Q} + b_{30} \frac{t}{Q} + b_{31} \frac{1}{Q} \ , \\
g_y^y &= b_{25} \ln Q + b_{32} \frac{t^2}{Q} + b_{33} \frac{t}{Q} + b_{34} \frac{1}{Q} \ ,
\end{aligned}
\tag{42}
$$

where, again, the values of the constants b_{25} through b_{34} are listed below in Sec. 10.

At this point the integrals in eqns. (37) can be calculated. Namely, substituting eqns. (41) into the second of eqns. (37), and performing the integrations, one obtains

$$
\begin{aligned}
A_{000} &= \frac{1}{L} \left[-b_1 I_2 + (Lb_1 - b_2) I_3 + Lb_2 I_4 - b_3 I_5 + (Lb_3 - b_4) I_6 + (Lb_4 - b_5) I_7 \right. \\
&\qquad \left. + (Lb_5 - b_6) I_8 + Lb_6 I_9 \right] \ , \\
A_{001} &= \frac{1}{L} \left[b_1 I_2 + b_2 I_3 + b_3 I_5 + b_4 I_6 + b_5 I_7 + b_6 I_8 \right] \ , \\
A_{010} &= \frac{1}{L} \left[-b_7 I_2 + (Lb_7 - b_8) I_3 + Lb_8 I_4 - b_9 I_5 + (Lb_9 - b_{10}) I_6 + (Lb_{10} - b_{11}) I_7 \right. \\
&\qquad \left. + (Lb_{11} - b_{12}) I_8 + Lb_{12} I_9 \right] \ , \\
A_{011} &= \frac{1}{L} \left[b_7 I_2 + b_8 I_3 + b_9 I_5 + b_{10} I_6 + b_{11} I_7 + b_{12} I_8 \right] \ , \\
A_{100} &= \frac{1}{L} \left[-b_{13} I_2 + (Lb_{13} - b_{14}) I_3 + Lb_{14} I_4 - b_{15} I_5 + (Lb_{15} - b_{16}) I_6 + (Lb_{16} - b_{17}) I_7 \right. \\
&\qquad \left. + (Lb_{17} - b_{18}) I_8 + Lb_{18} I_9 \right] \ , \\
A_{101} &= \frac{1}{L} \left[b_{13} I_2 + b_{14} I_3 + b_{15} I_5 + b_{16} I_6 + b_{17} I_7 + b_{18} I_8 \right] \ , \\
A_{110} &= \frac{1}{L} \left[-b_{19} I_2 + (Lb_{19} - b_{20}) I_3 + Lb_{20} I_4 - b_{21} I_5 + (Lb_{21} - b_{22}) I_6 + (Lb_{22} - b_{23}) I_7 \right. \\
&\qquad \left. + (Lb_{23} - b_{24}) I_8 + Lb_{24} I_9 \right] \ , \\
A_{111} &= \frac{1}{L} \left[b_{19} I_2 + b_{20} I_3 + b_{21} I_5 + b_{22} I_6 + b_{23} I_7 + b_{24} I_8 \right] \ ,
\end{aligned}
\tag{43}
$$

where $0 \equiv x$ and $1 \equiv y$ have been used for the first two subscripts of A_{kjI}. The indefinite forms $I_i(t)$ of the definite integrals I_i appearing in eqns. (43) are listed below in Sec. 13. For instance, $I_2 = I_2(L) - I_2(0)$, *etc.* Similarly, substituting eqns. (42) into the third of eqns. (37), and performing the integrations, one finds

$$B_{000} = \frac{1}{L}\,[\,-b_{25}I_{10} + Lb_{25}I_{11} - b_{26}I_1 + (Lb_{26} - b_{27})I_2 + (Lb_{27} - b_{28})I_3 + Lb_{28}I_4\,]\ ,$$
$$B_{001} = \frac{1}{L}\,[\,b_{25}I_{10} + b_{26}I_1 + b_{27}I_2 + b_{28}I_3\,]\ ,$$
$$B_{010} = B_{100} = \frac{1}{L}\,[\,-b_{29}I_1 + (Lb_{29} - b_{30})I_2 + (Lb_{30} - b_{31})I_3 + Lb_{31}I_4\,]\ , \qquad (44)$$
$$B_{011} = B_{101} = \frac{1}{L}\,[\,b_{29}I_1 + b_{30}I_2 + b_{31}I_3\,]\ ,$$
$$B_{110} = \frac{1}{L}\,[\,-b_{25}I_{10} + Lb_{25}I_{11} - b_{32}I_1 + (Lb_{32} - b_{33})I_2 + (Lb_{33} - b_{34})I_3 + Lb_{34}I_4\,]\ ,$$
$$B_{111} = \frac{1}{L}\,[\,b_{25}I_{10} + b_{32}I_1 + b_{33}I_2 + b_{34}I_3\,]\ ,$$

where the notation for the subscripts for B_{kjI} is the same as that used in eqns. (43), and again, the indefinite forms of the definite integrals I_i are listed in Sec. 13.

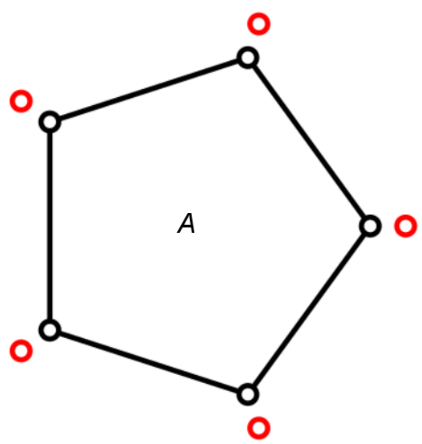

Figure 5. A boundary element discretization of the domain A.

Having defined all the terms in eqns. (37), the nodal displacements and tractions on the boundary are solved for by the following method. Figure 5 at left shows a domain A discretized with five boundary elements. The black lines and points in the figure represent the boundary elements. Now, putting the singularity $\mathbf{x}_0$ at each of the five red points, in turn, generates ten algebraic equations relating the twenty boundary displacements and tractions. At each black point, two of the boundary quantities are prescribed with the boundary conditions for the problem. Consequently, the resulting system of ten equations in ten unknows may then be solved for the remaining boundary quantities.

Once the nodal boundary displacements and tractions have been found, the displacements at any point in the interior of A may be found with the discretized form of eqn. (35), *i.e.,* with $c = 1$ for $\mathbf{x}_0$ in A,

$$u_k(\mathbf{x}_0) = -A_{kjI}u_j^I + B_{kjI}T_j^I\ . \qquad (45)$$

7. Gradients of the Green's Functions

Once all the displacements in the problem have been found, one turns attention to the calculation of the displacement gradients. With the notation $\partial f/\partial x_{0i} \equiv f_{:i}$, the gradient of eqn. (35) is

$$u_{k:m}(\mathbf{x}_0) = -\oint_t n_i G^k_{ij:m} u_j \mathrm{d}t + \oint_t g^k_{j:m} T_j \mathrm{d}t\ \ , \qquad (46)$$

where $c = 1$, *i.e.,* $\mathbf{x}_0$ is in A, or in discretized form,

$$u_{k:m}(\mathbf{x}_0) = -A_{kjI:m}u_j^I + B_{kjI:m}T_j^I\ , \qquad A_{kjI:m} = \int_0^L n_i G^k_{ij:m} f^I \mathrm{d}t\ \ , \qquad B_{kjI:m} = \int_0^L g^k_{j:m} f^I \mathrm{d}t\ \ . \qquad (47)$$

Thus, the evaluation of eqns. (47) requires the gradients of the Green's functions. So, using $r_{:x} = -X/r$

and $r_{:y} = -Y/r$, one obtains for the gradients of eqns. (28)

$$
\begin{aligned}
G^x_{xx:x} &= -\frac{1}{4\pi(1-\nu^*)}\left\{(3-2\nu^*)\left[-\frac{1}{r^2}+2\frac{X^2}{r^4}\right]+2\frac{Y^2}{r^4}-8\frac{X^2Y^2}{r^6}\right\}\,, \\
G^x_{xx:y} &= -\frac{1}{2\pi(1-\nu^*)}\left\{(5-2\nu^*)\frac{XY}{r^4}-4\frac{XY^3}{r^6}\right\}\,, \\
G^x_{xy:x} = G^x_{yx:x} &= -\frac{1}{2\pi(1-\nu^*)}\left\{(1-2\nu^*)\frac{XY}{r^4}+2\frac{X^3Y}{r^6}-2\frac{XY^3}{r^6}\right\}\,, \\
G^x_{xy:y} = G^x_{yx:y} &= -\frac{1}{4\pi(1-\nu^*)}\left\{-2(1-\nu^*)\frac{1}{r^2}+(7-4\nu^*)\frac{Y^2}{r^4}-\frac{X^2}{r^4}+4\frac{X^2Y^2}{r^6}-4\frac{Y^4}{r^6}\right\}\,, \\
G^x_{yy:x} &= \frac{1}{4\pi(1-\nu^*)}\left\{(1-2\nu^*)\left[-\frac{1}{r^2}+2\frac{X^2}{r^4}\right]+2\frac{Y^2}{r^4}-8\frac{X^2Y^2}{r^6}\right\}\,, \\
G^x_{yy:y} &= \frac{1}{2\pi(1-\nu^*)}\left\{(3-2\nu^*)\frac{XY}{r^4}-4\frac{XY^3}{r^6}\right\}\,, \\
G^y_{xx:x} &= \frac{1}{2\pi(1-\nu^*)}\left\{(3-2\nu^*)\frac{XY}{r^4}-4\frac{X^3Y}{r^6}\right\}\,, \\
G^y_{xx:y} &= \frac{1}{4\pi(1-\nu^*)}\left\{(1-2\nu^*)\left[-\frac{1}{r^2}+2\frac{Y^2}{r^4}\right]+2\frac{X^2}{r^4}-8\frac{X^2Y^2}{r^6}\right\}\,, \\
G^y_{xy:x} = G^y_{yx:x} &= -\frac{1}{4\pi(1-\nu^*)}\left\{-2(1-\nu^*)\frac{1}{r^2}+(7-4\nu^*)\frac{X^2}{r^4}-\frac{Y^2}{r^4}+4\frac{X^2Y^2}{r^6}-4\frac{X^4}{r^6}\right\}\,, \\
G^y_{xy:y} = G^y_{yx:y} &= -\frac{1}{2\pi(1-\nu^*)}\left\{(1-2\nu^*)\frac{XY}{r^4}+2\frac{XY^3}{r^6}-2\frac{X^3Y}{r^6}\right\}\,, \\
G^y_{yy:x} &= -\frac{1}{2\pi(1-\nu^*)}\left\{(5-2\nu^*)\frac{XY}{r^4}-4\frac{X^3Y}{r^6}\right\}\,, \\
G^y_{yy:y} &= -\frac{1}{4\pi(1-\nu^*)}\left\{(3-2\nu^*)\left[-\frac{1}{r^2}+2\frac{Y^2}{r^4}\right]+2\frac{X^2}{r^4}-8\frac{X^2Y^2}{r^6}\right\}\,,
\end{aligned} \tag{48}
$$

and for the gradients of eqns. (27)

$$
\begin{aligned}
g^x_{x:x} &= \frac{S}{4\pi(1-\nu^*)}\left\{(3-4\nu^*)\frac{X}{r^2}-2\frac{XY^2}{r^4}\right\}\,, \\
g^x_{x:y} &= \frac{S}{4\pi(1-\nu^*)}\left\{(5-4\nu^*)\frac{Y}{r^2}-2\frac{Y^3}{r^4}\right\}\,, \\
g^x_{y:x} = g^y_{x:x} &= \frac{S}{4\pi(1-\nu^*)}\left\{-\frac{Y}{r^2}+2\frac{X^2Y}{r^4}\right\}\,, \\
g^x_{y:y} = g^y_{x:y} &= \frac{S}{4\pi(1-\nu^*)}\left\{-\frac{X}{r^2}+2\frac{XY^2}{r^4}\right\}\,, \\
g^y_{y:x} &= \frac{S}{4\pi(1-\nu^*)}\left\{(5-4\nu^*)\frac{X}{r^2}-2\frac{X^3}{r^4}\right\}\,, \\
g^y_{y:y} &= \frac{S}{4\pi(1-\nu^*)}\left\{(3-4\nu^*)\frac{Y}{r^2}-2\frac{X^2Y}{r^4}\right\}\,.
\end{aligned} \tag{49}
$$

8. The Boundary Element Method – Displacement Gradient Solution

If one attempts to evaluate eqns. (47) when $\mathbf{x}_0$ on the boundary t, then the resulting integrals are divergent. Consequently, to calculate the displacement gradients $u_{i,j}$ at the boundary points requires a different procedure, which procedure is explained below in Sec. 8.2.

8.1. Displacement Gradients in A

Here eqns. (47) may be used to calculate the displacement gradients for points which are inside of the domain A. So, evaluating the gradients $G^k_{ij:m}$ as a function of t along a boundary element by substituting eqns. (38) and (39) into eqns. (48), one obtains the expressions

$$
\begin{aligned}
G^x_{xx:x} &= c_1\frac{1}{Q} + c_2\frac{t^2}{Q^2} + c_3\frac{t}{Q^2} + c_4\frac{1}{Q^2} + c_5\frac{t^4}{Q^3} + c_6\frac{t^3}{Q^3} + c_7\frac{t^2}{Q^3} + c_8\frac{t}{Q^3} + c_9\frac{1}{Q^3}\ ,\\
G^x_{xx:y} &= c_{10}\frac{t^2}{Q^2} + c_{11}\frac{t}{Q^2} + c_{12}\frac{1}{Q^2} + c_{13}\frac{t^4}{Q^3} + c_{14}\frac{t^3}{Q^3} + c_{15}\frac{t^2}{Q^3} + c_{16}\frac{t}{Q^3} + c_{17}\frac{1}{Q^3}\ ,\\
G^x_{xy:x} = G^x_{yx:x} &= c_{18}\frac{t^2}{Q^2} + c_{19}\frac{t}{Q^2} + c_{20}\frac{1}{Q^2} + c_{21}\frac{t^4}{Q^3} + c_{22}\frac{t^3}{Q^3} + c_{23}\frac{t^2}{Q^3} + c_{24}\frac{t}{Q^3} + c_{25}\frac{1}{Q^3}\ ,\\
G^x_{xy:y} = G^x_{yx:y} &= c_{26}\frac{1}{Q} + c_{27}\frac{t^2}{Q^2} + c_{28}\frac{t}{Q^2} + c_{29}\frac{1}{Q^2} + c_{30}\frac{t^4}{Q^3} + c_{31}\frac{t^3}{Q^3} + c_{32}\frac{t^2}{Q^3} + c_{33}\frac{t}{Q^3} + c_{34}\frac{1}{Q^3}\ ,\\
G^x_{yy:x} &= c_{35}\frac{1}{Q} + c_{36}\frac{t^2}{Q^2} + c_{37}\frac{t}{Q^2} + c_{38}\frac{1}{Q^2} - c_5\frac{t^4}{Q^3} - c_6\frac{t^3}{Q^3} - c_7\frac{t^2}{Q^3} - c_8\frac{t}{Q^3} - c_9\frac{1}{Q^3}\ ,\\
G^x_{yy:y} &= c_{39}\frac{t^2}{Q^2} + c_{40}\frac{t}{Q^2} + c_{41}\frac{1}{Q^2} - c_{13}\frac{t^4}{Q^3} - c_{14}\frac{t^3}{Q^3} - c_{15}\frac{t^2}{Q^3} - c_{16}\frac{t}{Q^3} - c_{17}\frac{1}{Q^3}\ ,\\
G^y_{xx:x} &= c_{39}\frac{t^2}{Q^2} + c_{40}\frac{t}{Q^2} + c_{41}\frac{1}{Q^2} + c_{42}\frac{t^4}{Q^3} + c_{43}\frac{t^3}{Q^3} + c_{44}\frac{t^2}{Q^3} + c_{45}\frac{t}{Q^3} + c_{46}\frac{1}{Q^3}\ ,\\
G^y_{xx:y} &= c_{35}\frac{1}{Q} + c_{47}\frac{t^2}{Q^2} + c_{48}\frac{t}{Q^2} + c_{49}\frac{1}{Q^2} - c_5\frac{t^4}{Q^3} - c_6\frac{t^3}{Q^3} - c_7\frac{t^2}{Q^3} - c_8\frac{t}{Q^3} - c_9\frac{1}{Q^3}\ ,\\
G^y_{xy:x} = G^y_{yx:x} &= c_{26}\frac{1}{Q} + c_{50}\frac{t^2}{Q^2} + c_{51}\frac{t}{Q^2} + c_{52}\frac{1}{Q^2} + c_{53}\frac{t^4}{Q^3} + c_{54}\frac{t^3}{Q^3} + c_{55}\frac{t^2}{Q^3} + c_{56}\frac{t}{Q^3} + c_{57}\frac{1}{Q^3}\ ,\\
G^y_{xy:y} = G^y_{yx:y} &= c_{18}\frac{t^2}{Q^2} + c_{19}\frac{t}{Q^2} + c_{20}\frac{1}{Q^2} - c_{21}\frac{t^4}{Q^3} - c_{22}\frac{t^3}{Q^3} - c_{23}\frac{t^2}{Q^3} - c_{24}\frac{t}{Q^3} - c_{25}\frac{1}{Q^3}\ ,\\
G^y_{yy:x} &= c_{10}\frac{t^2}{Q^2} + c_{11}\frac{t}{Q^2} + c_{12}\frac{1}{Q^2} - c_{42}\frac{t^4}{Q^3} - c_{43}\frac{t^3}{Q^3} - c_{44}\frac{t^2}{Q^3} - c_{45}\frac{t}{Q^3} - c_{46}\frac{1}{Q^3}\ ,\\
G^y_{yy:y} &= c_1\frac{1}{Q} + c_{58}\frac{t^2}{Q^2} + c_{59}\frac{t}{Q^2} + c_{60}\frac{1}{Q^2} + c_5\frac{t^4}{Q^3} + c_6\frac{t^3}{Q^3} + c_7\frac{t^2}{Q^3} + c_8\frac{t}{Q^3} + c_9\frac{1}{Q^3}\ ,
\end{aligned}
\tag{50}
$$

where the values of the constants c_1 through c_{60} are listed below in Sec. 11. Next, performing the dot product $n_i G^k_{ij:m}$ one has, where the values of the constants d_1 through d_{72} are listed below in Sec. 12,

$$
\begin{aligned}
n_i G^x_{ix:x} &= d_1\frac{1}{Q} + d_2\frac{t^2}{Q^2} + d_3\frac{t}{Q^2} + d_4\frac{1}{Q^2} + d_5\frac{t^4}{Q^3} + d_6\frac{t^3}{Q^3} + d_7\frac{t^2}{Q^3} + d_8\frac{t}{Q^3} + d_9\frac{1}{Q^3}\ ,\\
n_i G^x_{ix:y} &= d_{10}\frac{1}{Q} + d_{11}\frac{t^2}{Q^2} + d_{12}\frac{t}{Q^2} + d_{13}\frac{1}{Q^2} + d_{14}\frac{t^4}{Q^3} + d_{15}\frac{t^3}{Q^3} + d_{16}\frac{t^2}{Q^3} + d_{17}\frac{t}{Q^3} + d_{18}\frac{1}{Q^3}\ ,\\
n_i G^x_{iy:x} &= d_{19}\frac{1}{Q} + d_{20}\frac{t^2}{Q^2} + d_{21}\frac{t}{Q^2} + d_{22}\frac{1}{Q^2} + d_{23}\frac{t^4}{Q^3} + d_{24}\frac{t^3}{Q^3} + d_{25}\frac{t^2}{Q^3} + d_{26}\frac{t}{Q^3} + d_{27}\frac{1}{Q^3}\ ,
\end{aligned}
$$

$$n_i G^x_{iy:y} = d_{28}\frac{1}{Q} + d_{29}\frac{t^2}{Q^2} + d_{30}\frac{t}{Q^2} + d_{31}\frac{1}{Q^2} + d_{32}\frac{t^4}{Q^3} + d_{33}\frac{t^3}{Q^3} + d_{34}\frac{t^2}{Q^3} + d_{35}\frac{t}{Q^3} + d_{36}\frac{1}{Q^3} \ , \quad (51)$$

$$n_i G^y_{ix:x} = d_{37}\frac{1}{Q} + d_{38}\frac{t^2}{Q^2} + d_{39}\frac{t}{Q^2} + d_{40}\frac{1}{Q^2} + d_{41}\frac{t^4}{Q^3} + d_{42}\frac{t^3}{Q^3} + d_{43}\frac{t^2}{Q^3} + d_{44}\frac{t}{Q^3} + d_{45}\frac{1}{Q^3} \ ,$$

$$n_i G^y_{ix:y} = d_{46}\frac{1}{Q} + d_{47}\frac{t^2}{Q^2} + d_{48}\frac{t}{Q^2} + d_{49}\frac{1}{Q^2} + d_{50}\frac{t^4}{Q^3} + d_{51}\frac{t^3}{Q^3} + d_{52}\frac{t^2}{Q^3} + d_{53}\frac{t}{Q^3} + d_{54}\frac{1}{Q^3} \ ,$$

$$n_i G^y_{iy:x} = d_{55}\frac{1}{Q} + d_{56}\frac{t^2}{Q^2} + d_{57}\frac{t}{Q^2} + d_{58}\frac{1}{Q^2} + d_{59}\frac{t^4}{Q^3} + d_{60}\frac{t^3}{Q^3} + d_{61}\frac{t^2}{Q^3} + d_{62}\frac{t}{Q^3} + d_{63}\frac{1}{Q^3} \ ,$$

$$n_i G^y_{iy:y} = d_{64}\frac{1}{Q} + d_{65}\frac{t^2}{Q^2} + d_{66}\frac{t}{Q^2} + d_{67}\frac{1}{Q^2} + d_{68}\frac{t^4}{Q^3} + d_{69}\frac{t^3}{Q^3} + d_{70}\frac{t^2}{Q^3} + d_{71}\frac{t}{Q^3} + d_{72}\frac{1}{Q^3} \ ,$$

and then substitute eqns. (38) and (39) into eqns. (49) to obtain, again, where the constants d_{73} through d_{100} are listed below in Sec. 12,

$$\begin{aligned}
g^x_{x:x} &= d_{73}\frac{t}{Q} + d_{74}\frac{1}{Q} + d_{75}\frac{t^3}{Q^2} + d_{76}\frac{t^2}{Q^2} + d_{77}\frac{t}{Q^2} + d_{78}\frac{1}{Q^2} \ , \\
g^x_{x:y} &= d_{79}\frac{t}{Q} + d_{80}\frac{1}{Q} + d_{81}\frac{t^3}{Q^2} + d_{82}\frac{t^2}{Q^2} + d_{83}\frac{t}{Q^2} + d_{84}\frac{1}{Q^2} \ , \\
g^x_{y:x} = g^y_{x:x} &= d_{85}\frac{t}{Q} + d_{86}\frac{1}{Q} + d_{87}\frac{t^3}{Q^2} + d_{88}\frac{t^2}{Q^2} + d_{89}\frac{t}{Q^2} + d_{90}\frac{1}{Q^2} \ , \\
g^x_{y:y} = g^y_{x:y} &= d_{91}\frac{t}{Q} + d_{92}\frac{1}{Q} - d_{75}\frac{t^3}{Q^2} - d_{76}\frac{t^2}{Q^2} - d_{77}\frac{t}{Q^2} - d_{78}\frac{1}{Q^2} \ , \\
g^y_{y:x} &= d_{93}\frac{t}{Q} + d_{94}\frac{1}{Q} + d_{95}\frac{t^3}{Q^2} + d_{96}\frac{t^2}{Q^2} + d_{97}\frac{t}{Q^2} + d_{98}\frac{1}{Q^2} \ , \\
g^y_{y:y} &= d_{99}\frac{t}{Q} + d_{100}\frac{1}{Q} - d_{87}\frac{t^3}{Q^2} - d_{88}\frac{t^2}{Q^2} - d_{89}\frac{t}{Q^2} - d_{90}\frac{1}{Q^2} \ .
\end{aligned} \quad (52)$$

At this point, the integrals (47) may be evaluated. Consequently, substitution of eqns. (51) into the second of eqns. (47), and performing the integrations gives

$$\begin{aligned}
A_{000:0} &= \frac{1}{L}\,[\,-d_1 I_3 + L d_1 I_4 - d_2 I_6 + (L d_2 - d_3) I_7 + (L d_3 - d_4) I_8 + L d_4 I_9 - d_5 I_{12} \\
&\quad + (L d_5 - d_6) I_{13} + (L d_6 - d_7) I_{14} + (L d_7 - d_8) I_{15} + (L d_8 - d_9) I_{16} + L d_9 I_{17}\,] \ , \\
A_{001:0} &= \frac{1}{L}\,[\,d_1 I_3 + d_2 I_6 + d_3 I_7 + d_4 I_8 + d_5 I_{12} + d_6 I_{13} + d_7 I_{14} + d_8 I_{15} + d_9 I_{16}\,] \ , \\
A_{000:1} &= \frac{1}{L}\,[\,-d_{10} I_3 + L d_{10} I_4 - d_{11} I_6 + (L d_{11} - d_{12}) I_7 + (L d_{12} - d_{13}) I_8 + L d_{13} I_9 - d_{14} I_{12} \\
&\quad + (L d_{14} - d_{15}) I_{13} + (L d_{15} - d_{16}) I_{14} + (L d_{16} - d_{17}) I_{15} + (L d_{17} - d_{18}) I_{16} + L d_{18} I_{17}\,] \ , \\
A_{001:1} &= \frac{1}{L}\,[\,d_{10} I_3 + d_{11} I_6 + d_{12} I_7 + d_{13} I_8 + d_{14} I_{12} + d_{15} I_{13} + d_{16} I_{14} + d_{17} I_{15} + d_{18} I_{16}\,] \ , \\
A_{010:0} &= \frac{1}{L}\,[\,-d_{19} I_3 + L d_{19} I_4 - d_{20} I_6 + (L d_{20} - d_{21}) I_7 + (L d_{21} - d_{22}) I_8 + L d_{22} I_9 - d_{23} I_{12} \\
&\quad + (L d_{23} - d_{24}) I_{13} + (L d_{24} - d_{25}) I_{14} + (L d_{25} - d_{26}) I_{15} + (L d_{26} - d_{27}) I_{16} + L d_{27} I_{17}\,] \ , \\
A_{011:0} &= \frac{1}{L}\,[\,d_{19} I_3 + d_{20} I_6 + d_{21} I_7 + d_{22} I_8 + d_{23} I_{12} + d_{24} I_{13} + d_{25} I_{14} + d_{26} I_{15} + d_{27} I_{16}\,] \ , \\
A_{010:1} &= \frac{1}{L}\,[\,-d_{28} I_3 + L d_{28} I_4 - d_{29} I_6 + (L d_{29} - d_{30}) I_7 + (L d_{30} - d_{31}) I_8 + L d_{31} I_9 - d_{32} I_{12} \\
&\quad + (L d_{32} - d_{33}) I_{13} + (L d_{33} - d_{34}) I_{14} + (L d_{34} - d_{35}) I_{15} + (L d_{35} - d_{36}) I_{16} + L d_{36} I_{17}\,] \ , \\
A_{011:1} &= \frac{1}{L}\,[\,d_{28} I_3 + d_{29} I_6 + d_{30} I_7 + d_{31} I_8 + d_{32} I_{12} + d_{33} I_{13} + d_{34} I_{14} + d_{35} I_{15} + d_{36} I_{16}\,] \ ,
\end{aligned} \quad (53)$$

$$A_{100:0} = \frac{1}{L}\,[\,-d_{37}I_3 + Ld_{37}I_4 - d_{38}I_6 + (Ld_{38} - d_{39})I_7 + (Ld_{39} - d_{40})I_8 + Ld_{40}I_9 - d_{41}I_{12} + (Ld_{41} - d_{42})I_{13} + (Ld_{42} - d_{43})I_{14} + (Ld_{43} - d_{44})I_{15} + (Ld_{44} - d_{45})I_{16} + Ld_{45}I_{17}\,]\ ,$$

$$A_{101:0} = \frac{1}{L}\,[\,d_{37}I_3 + d_{38}I_6 + d_{39}I_7 + d_{40}I_8 + d_{41}I_{12} + d_{42}I_{13} + d_{43}I_{14} + d_{44}I_{15} + d_{45}I_{16}\,]\ ,$$

$$A_{100:1} = \frac{1}{L}\,[\,-d_{46}I_3 + Ld_{46}I_4 - d_{47}I_6 + (Ld_{47} - d_{48})I_7 + (Ld_{48} - d_{49})I_8 + Ld_{49}I_9 - d_{50}I_{12} + (Ld_{50} - d_{51})I_{13} + (Ld_{51} - d_{52})I_{14} + (Ld_{52} - d_{53})I_{15} + (Ld_{53} - d_{54})I_{16} + Ld_{54}I_{17}\,]\ ,$$

$$A_{101:1} = \frac{1}{L}\,[\,d_{46}I_3 + d_{47}I_6 + d_{48}I_7 + d_{49}I_8 + d_{50}I_{12} + d_{51}I_{13} + d_{52}I_{14} + d_{53}I_{15} + d_{54}I_{16}\,]\ ,$$

$$A_{110:0} = \frac{1}{L}\,[\,-d_{55}I_3 + Ld_{55}I_4 - d_{56}I_6 + (Ld_{56} - d_{57})I_7 + (Ld_{57} - d_{58})I_8 + Ld_{58}I_9 - d_{59}I_{12} + (Ld_{59} - d_{60})I_{13} + (Ld_{60} - d_{61})I_{14} + (Ld_{61} - d_{62})I_{15} + (Ld_{62} - d_{63})I_{16} + Ld_{63}I_{17}\,]\ ,$$

$$A_{111:0} = \frac{1}{L}\,[\,d_{55}I_3 + d_{56}I_6 + d_{57}I_7 + d_{58}I_8 + d_{59}I_{12} + d_{60}I_{13} + d_{61}I_{14} + d_{62}I_{15} + d_{63}I_{16}\,]\ ,$$

$$A_{110:1} = \frac{1}{L}\,[\,-d_{64}I_3 + Ld_{64}I_4 - d_{65}I_6 + (Ld_{65} - d_{66})I_7 + (Ld_{66} - d_{67})I_8 + Ld_{67}I_9 - d_{68}I_{12} + (Ld_{68} - d_{69})I_{13} + (Ld_{69} - d_{70})I_{14} + (Ld_{70} - d_{71})I_{15} + (Ld_{71} - d_{72})I_{16} + Ld_{72}I_{17}\,]\ ,$$

$$A_{111:1} = \frac{1}{L}\,[\,d_{64}I_3 + d_{65}I_6 + d_{66}I_7 + d_{67}I_8 + d_{68}I_{12} + d_{69}I_{13} + d_{70}I_{14} + d_{71}I_{15} + d_{72}I_{16}\,]\ ,$$

where, as in Sec. 6, the indefinite integrals $I_i(t)$ are listed below in Sec. 13, *e.g.,* for I_3 above, $I_3 = I_3(L) - I_3(0)$. Also, for the first, second and fourth indices of $A_{kjl:m}$, the notation $0 \equiv x$ and $1 \equiv y$ has been used. Similarly, substitution of eqns. (52) into the third of eqns. (47), and performing the integrations yields

$$\begin{aligned}
B_{000:0} &= \frac{1}{L}\,[\,-d_{73}I_2 + (Ld_{73} - d_{74})I_3 + Ld_{74}I_4 - d_{75}I_5 + (Ld_{75} - d_{76})I_6 + (Ld_{76} - d_{77})I_7 + (Ld_{77} - d_{78})I_8 + Ld_{78}I_9\,]\ ,\\
B_{001:0} &= \frac{1}{L}\,[\,d_{73}I_2 + d_{74}I_3 + d_{75}I_5 + d_{76}I_6 + d_{77}I_7 + d_{78}I_8\,]\ ,\\
B_{000:1} &= \frac{1}{L}\,[\,-d_{79}I_2 + (Ld_{79} - d_{80})I_3 + Ld_{80}I_4 - d_{81}I_5 + (Ld_{81} - d_{82})I_6 + (Ld_{82} - d_{83})I_7 + (Ld_{83} - d_{84})I_8 + Ld_{84}I_9\,]\ ,\\
B_{001:1} &= \frac{1}{L}\,[\,d_{79}I_2 + d_{80}I_3 + d_{81}I_5 + d_{82}I_6 + d_{83}I_7 + d_{84}I_8\,]\ ,\\
B_{010:0} = B_{100:0} &= \frac{1}{L}\,[\,-d_{85}I_2 + (Ld_{85} - d_{86})I_3 + Ld_{86}I_4 - d_{87}I_5 + (Ld_{87} - d_{88})I_6 + (Ld_{88} - d_{89})I_7 + (Ld_{89} - d_{90})I_8 + Ld_{90}I_9\,]\ ,\\
B_{011:0} = B_{101:0} &= \frac{1}{L}\,[\,d_{85}I_2 + d_{86}I_3 + d_{87}I_5 + d_{88}I_6 + d_{89}I_7 + d_{90}I_8\,]\ ,\\
B_{010:1} = B_{100:1} &= \frac{1}{L}\,[\,-d_{91}I_2 + (Ld_{91} - d_{92})I_3 + Ld_{92}I_4 + d_{75}I_5 - (Ld_{75} - d_{76})I_6 - (Ld_{76} - d_{77})I_7 - (Ld_{77} - d_{78})I_8 - Ld_{78}I_9\,]\ ,\\
B_{011:1} = B_{101:1} &= \frac{1}{L}\,[\,d_{91}I_2 + d_{92}I_3 - d_{75}I_5 - d_{76}I_6 - d_{77}I_7 - d_{78}I_8\,]\ ,\\
B_{110:0} &= \frac{1}{L}\,[\,-d_{93}I_2 + (Ld_{93} - d_{94})I_3 + Ld_{94}I_4 - d_{95}I_5 + (Ld_{95} - d_{96})I_6 + (Ld_{96} - d_{97})I_7 + (Ld_{97} - d_{98})I_8 + Ld_{98}I_9\,]\ ,\\
B_{111:0} &= \frac{1}{L}\,[\,d_{93}I_2 + d_{94}I_3 + d_{95}I_5 + d_{96}I_6 + d_{97}I_7 + d_{98}I_8\,]\ ,
\end{aligned} \tag{54}$$

$$B_{110:1} = \frac{1}{L}\ [-d_{99}I_2 + (Ld_{99} - d_{100})I_3 + Ld_{100}I_4 + d_{87}I_5 - (Ld_{87} - d_{88})I_6$$
$$-(Ld_{88} - d_{89})I_7 - (Ld_{89} - d_{90})I_8 - Ld_{90}I_9\ ]\ ,$$
$$B_{111:1} = \frac{1}{L}\ [\ d_{99}I_2 + d_{100}I_3 - d_{87}I_5 - d_{88}I_6 - d_{89}I_7 - d_{90}I_8\ ]\ ,$$

where the notation used for the indices of $B_{kjI:m}$ is the same as that used in eqns. (53), and again, the indefinite forms of the definite integrals I_i are listed below in Sec. 13. In any case, knowing the displacement gradients from eqn. (47), the strain components are known via $\varepsilon_{ij} = 1/2\,(u_{i,j} + u_{j,i})$.

8.2. Strain Components on t

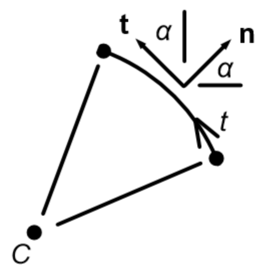

Figure 6. A curved portion of the boundary t.

While the boundary element of Sec. 6 is straight, it does make sense to consider the curvature of the boundary when calculating the strains, and thus the stresses, on the boundary. At left, in Fig. 6, is shown a curved section of the boundary. The outward pointing unit normal vector is given by

$$\mathbf{n} = \cos\alpha\, \mathbf{e}_x + \sin\alpha\, \mathbf{e}_y\ , \tag{55}$$

where α is the orientation of a differential of length on the boundary, and the point C is the center of curvature. The curvature κ is the tangential derivative of α, *i.e.,*

$$\kappa = \alpha_{,t}\ . \tag{56}$$

Note that κ may be positive or negative. Notwithstanding, in the curvilinear nt–system of the boundary, the strain components are

$$\varepsilon_{nn} = u_{n,n}\ , \qquad \varepsilon_{tt} = u_{t,t} + \kappa u_n\ , \qquad \varepsilon_{nt} = \frac{1}{2}\left(u_{n,t} + u_{t,n} - \kappa u_t\right). \tag{57}$$

One notices the similarity of eqns. (57) to the strain components in polar coordinates, which are,

$$\varepsilon_{rr} = u_{r,r}\ , \qquad \varepsilon_{\theta\theta} = \frac{1}{r}u_{\theta,\theta} + \frac{1}{r}u_r\ , \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}u_{r,\theta} + u_{\theta,r} - \frac{1}{r}u_\theta\right). \tag{58}$$

In other words, α is like θ, and

$$\frac{\partial}{\partial t} = \frac{1}{r}\frac{\partial}{\partial \alpha}\ , \qquad \kappa = \frac{1}{r}\ . \tag{59}$$

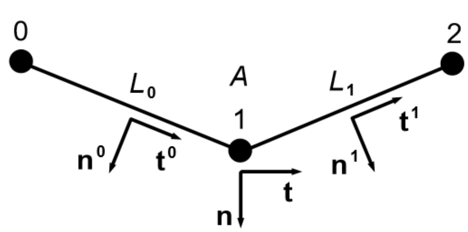

Figure 7. Two boundary elements with the point 1 being on a smooth portion of the boundary.

To calculate κ at the nodes on the boundary, the situation depicted in Fig. 7 at left is employed. The boundary is taken as being smooth at the point 1. The unit tangent and normal vectors $\mathbf{t}$ and $\mathbf{n}$ at point 1 are calculated as

$$\mathbf{t} = \frac{\mathbf{t}^0 + \mathbf{t}^1}{|\ \mathbf{t}^0 + \mathbf{t}^1\ |}\ , \qquad \mathbf{n} = \frac{\mathbf{n}^0 + \mathbf{n}^1}{|\ \mathbf{n}^0 + \mathbf{n}^1\ |}\ . \tag{60}$$

In any case, having the element values α^I $(I = 0{,}1)$ pictured below in Fig. 8, which are calculated via

$$\alpha^I = \tan^{-1}\left(\frac{n_y^I}{n_x^I}\right), \tag{61}$$

the derivative (56) at point 1 is calculated via finite difference, *viz.,*

$$\kappa^1 = \frac{\alpha^1 - \alpha^0}{\frac{1}{2}(L_0 + L_1)} . \tag{62}$$

If point 0 is on a corner, then $\kappa^0 = \kappa^1$, and if point 2 is on a corner, then $\kappa^2 = \kappa^1$.

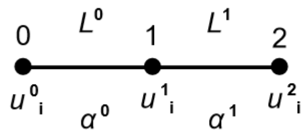

Figure 8. Element quantities corresponding to the two boundary elements in Fig. 7.

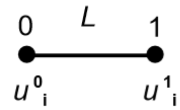

Figure 9. A single boundary element.

Now, the transformation matrix ψ_{ij} at the boundary nodes is

$$\psi_{ij} = \begin{bmatrix} n_x & n_y \\ t_x & t_y \end{bmatrix}, \qquad v_i^{\mathrm{nt}} = \psi_{ij} v_j^{\mathrm{xy}} , \tag{63}$$

where the components of $\mathbf{n}$ and $\mathbf{t}$ are calculated with eqns. (60), or with $\mathbf{n}^0$ and $\mathbf{t}^0$ if point 0 is on a corner, or with $\mathbf{n}^1$ and $\mathbf{t}^1$ if point 2 is on a corner. In the second of eqns. (63), v_i^{nt} are the components of a vector $\mathbf{v}$ in the nt–system, and v_j^{xy} are the components of that vector in the xy–system.

The procedure for calculating the strains is, first, transform the nodal vectors u_i^I and T_i^I from the boundary solution to the nt–system via eqns. (63). Next, the tangential derivatives for each element, pictured in Fig. 9, are calculated, again, with finite difference

$$u_{i,t}^{\mathrm{e}} = \frac{u_i^1 - u_i^0}{L} , \qquad (i = n, t) , \tag{64}$$

where e is the element number, and then the tangential derivatives at point 1 in Fig. 8 are calculated by averaging the abutting element values

$$u_{i,t}^1 = \frac{1}{2}\left(u_{i,t}^{\mathrm{e0}} + u_{i,t}^{\mathrm{e1}}\right) . \tag{65}$$

If point 0 in Fig. 8 is on a corner, then $u_{i,t}^0 = u_{i,t}^{\mathrm{e0}}$, and if point 2 in Fig. 8 is on a corner, then $u_{i,t}^2 = u_{i,t}^{\mathrm{e1}}$.

At this point, $u_{n,t}$ and $u_{t,t}$ are known at all the boundary nodes. To find $u_{n,n}$ and $u_{t,n}$ look at Hooke's Law

$$\sigma_{nn} = \frac{1}{S(1-2\nu^*)}\left[\, (1-\nu^*)\varepsilon_{nn} + \nu^*\varepsilon_{tt} \,\right], \qquad \sigma_{nt} = \frac{1}{S}\varepsilon_{nt} , \tag{66}$$

cf., eqns. (4). Putting $T_n = \sigma_{nn}$ and $T_t = \sigma_{nt}$ into eqns. (66), and substituting eqns. (57) into eqns. (66), some minor algebra yields

$$u_{n,n} = \frac{1}{1-\nu^*}\left[\, S(1-2\nu^*)T_n - \nu^* u_{t,t} - \nu^*\kappa u_n \,\right], \qquad u_{t,n} = 2ST_t - u_{n,t} + \kappa u_t , \tag{67}$$

so that now $u_{n,n}$, $u_{n,t}$, $u_{t,n}$ and $u_{t,t}$ are known at all the boundary nodes. Equations (57) then give the strain components in the nt–system, which are transformed back to the xy–system via

$$\varepsilon_{kl}^{\mathrm{xy}} = \varepsilon_{ij}^{\mathrm{nt}} \psi_{ik} \psi_{jl} \, . \tag{68}$$

9. Values of the Constants a_i

The values of the constants a_i appearing in eqns. (40) are

$$a_1 = -\frac{(3-2\nu^*)}{4\pi(1-\nu^*)}\, t_x \,, \qquad a_2 = -\frac{(3-2\nu^*)}{4\pi(1-\nu^*)}\, r_{0x} \,, \qquad a_3 = \frac{1}{2\pi(1-\nu^*)}\, t_x t_y^2 \,,$$

$$a_4 = \frac{1}{2\pi(1-\nu^*)}\, t_y\left(2t_x r_{0y} + t_y r_{0x}\right) \,, \qquad a_5 = \frac{1}{2\pi(1-\nu^*)}\left(t_x r_{0y} + 2t_y r_{0x}\right) r_{0y} \,,$$

$$a_6 = \frac{1}{2\pi(1-\nu^*)}\, r_{0x} r_{0y}^2 \,, \qquad a_7 = -\frac{1}{2\pi}\, t_y \,, \qquad a_8 = -\frac{1}{2\pi}\, r_{0y} \,,$$

$$a_9 = -\frac{1}{4\pi(1-\nu^*)}\, t_y\left(t_x^2 - t_y^2\right) \,, \qquad a_{10} = -\frac{1}{4\pi(1-\nu^*)}\left[\, t_x\left(t_x r_{0y} + 2t_y r_{0x}\right) - 3t_y^2 r_{0y} \,\right] \,,$$

$$a_{11} = -\frac{1}{4\pi(1-\nu^*)}\left[\,\left(2t_x r_{0y} + t_y r_{0x}\right) r_{0x} - 3t_y r_{0y}^2 \,\right] \,, \qquad a_{12} = -\frac{1}{4\pi(1-\nu^*)}\left(r_{0x}^2 - r_{0y}^2\right) r_{0y} \,,$$

$$a_{13} = \frac{(1-2\nu^*)}{4\pi(1-\nu^*)}\, t_x \,, \qquad a_{14} = \frac{(1-2\nu^*)}{4\pi(1-\nu^*)}\, r_{0x} \,, \qquad a_{15} = \frac{(1-2\nu^*)}{4\pi(1-\nu^*)}\, t_y \,,$$

$$a_{16} = \frac{(1-2\nu^*)}{4\pi(1-\nu^*)}\, r_{0y} \,, \qquad a_{17} = -\frac{1}{2\pi(1-\nu^*)}\, t_x^2 t_y \,, \qquad a_{18} = -\frac{1}{2\pi(1-\nu^*)}\, t_x\left(t_x r_{0y} + 2t_y r_{0x}\right) \,,$$

$$a_{19} = -\frac{1}{2\pi(1-\nu^*)}\left(2t_x r_{0y} + t_y r_{0x}\right) r_{0x} \,, \qquad a_{20} = -\frac{1}{2\pi(1-\nu^*)}\, r_{0x}^2 r_{0y} \,, \qquad a_{21} = -\frac{1}{2\pi}\, t_x \,,$$

$$a_{22} = -\frac{1}{2\pi}\, r_{0x} \,, \qquad a_{23} = -\frac{1}{4\pi(1-\nu^*)}\, t_x\left(t_y^2 - t_x^2\right) \,,$$

$$a_{24} = -\frac{1}{4\pi(1-\nu^*)}\left[\, t_y\left(2t_x r_{0y} + t_y r_{0x}\right) - 3t_x^2 r_{0x} \,\right] \,,$$

$$a_{25} = -\frac{1}{4\pi(1-\nu^*)}\left[\,\left(t_x r_{0y} + 2t_y r_{0x}\right) r_{0y} - 3t_x r_{0x}^2 \,\right] \,, \qquad a_{26} = -\frac{1}{4\pi(1-\nu^*)}\left(r_{0y}^2 - r_{0x}^2\right) r_{0x} \,,$$

$$a_{27} = -\frac{(3-2\nu^*)}{4\pi(1-\nu^*)}\, t_y \,, \qquad a_{28} = -\frac{(3-2\nu^*)}{4\pi(1-\nu^*)}\, r_{0y} \,.$$

10. Values of the Constants b_i

The values of the constants b_i appearing in eqns. (41) through (44) are listed below. Note that the values of the constants a_i are listed directly above in Sec. 9.

$$b_1 = n_x a_1 + n_y a_7 \,, \qquad b_2 = n_x a_2 + n_y a_8 \,, \qquad b_3 = n_x a_3 + n_y a_9 \,,$$

$$b_4 = n_x a_4 + n_y a_{10} \,, \qquad b_5 = n_x a_5 + n_y a_{11} \,, \qquad b_6 = n_x a_6 + n_y a_{12} \,,$$

$$b_7 = n_x a_7 + n_y a_{13} \,, \qquad b_8 = n_x a_8 + n_y a_{14} \,, \qquad b_9 = n_x a_9 - n_y a_3 \,,$$

$$b_{10} = n_x a_{10} - n_y a_4 \,, \qquad b_{11} = n_x a_{11} - n_y a_5 \,, \qquad b_{12} = n_x a_{12} - n_y a_6 \,,$$

$$b_{13} = n_x a_{15} + n_y a_{21} \,, \qquad b_{14} = n_x a_{16} + n_y a_{22} \,, \qquad b_{15} = n_x a_{17} + n_y a_{23} \,,$$

$$b_{16} = n_x a_{18} + n_y a_{24} \,, \qquad b_{17} = n_x a_{19} + n_y a_{25} \,, \qquad b_{18} = n_x a_{20} + n_y a_{26} \,,$$

$$b_{19} = n_x a_{21} + n_y a_{27} \,, \qquad b_{20} = n_x a_{22} + n_y a_{28} \,, \qquad b_{21} = n_x a_{23} - n_y a_{17} \,,$$

$$b_{22} = n_x a_{24} - n_y a_{18} \,, \qquad b_{23} = n_x a_{25} - n_y a_{19} \,, \qquad b_{24} = n_x a_{26} - n_y a_{20} \,,$$

$$b_{25} = -\frac{(3-4\nu^*)S}{8\pi(1-\nu^*)} \,, \qquad b_{26} = -\frac{S}{4\pi(1-\nu^*)}\, t_y^2 \,, \qquad b_{27} = -\frac{S}{2\pi(1-\nu^*)}\, t_y r_{0y} \,,$$

$$b_{28} = -\frac{S}{4\pi(1-\nu^*)}\, r_{0y}^2 \ , \qquad b_{29} = \frac{S}{4\pi(1-\nu^*)}\, t_x t_y \ , \qquad b_{30} = \frac{S}{4\pi(1-\nu^*)}\, (t_x r_{0y} + t_y r_{0x}) \ ,$$
$$b_{31} = \frac{S}{4\pi(1-\nu^*)}\, r_{0x} r_{0y} \ , \qquad b_{32} = -\frac{S}{4\pi(1-\nu^*)}\, t_x^2 \ , \qquad b_{33} = -\frac{S}{2\pi(1-\nu^*)}\, t_x r_{0x} \ ,$$
$$b_{34} = -\frac{S}{4\pi(1-\nu^*)}\, r_{0x}^2 \ .$$

11. Values of the Constants c_i

The values of the constants c_i appearing in eqns. (50) are listed below.

$$c_1 = \frac{(3-2\nu^*)}{4\pi(1-\nu^*)} \ , \qquad c_2 = -\frac{1}{2\pi(1-\nu^*)}\,[\,(3-2\nu^*)t_x^2 + t_y^2\,] \ ,$$
$$c_3 = -\frac{1}{\pi(1-\nu^*)}\,[\,(3-2\nu^*)t_x r_{0x} + t_y r_{0y}\,] \ , \qquad c_4 = -\frac{1}{2\pi(1-\nu^*)}\,[\,(3-2\nu^*)r_{0x}^2 + r_{0y}^2\,] \ ,$$
$$c_5 = \frac{2}{\pi(1-\nu^*)}\, t_x^2 t_y^2 \ , \qquad c_6 = \frac{4}{\pi(1-\nu^*)}\, t_x t_y (t_x r_{0y} + t_y r_{0x}) \ ,$$
$$c_7 = \frac{2}{\pi(1-\nu^*)}\,(t_x^2 r_{0y}^2 + 4 t_x t_y r_{0x} r_{0y} + t_y^2 r_{0x}^2) \ , \qquad c_8 = \frac{4}{\pi(1-\nu^*)}\,(t_x r_{0y} + t_y r_{0x}) r_{0x} r_{0y} \ ,$$
$$c_9 = \frac{2}{\pi(1-\nu^*)}\, r_{0x}^2 r_{0y}^2 \ , \qquad c_{10} = -\frac{(5-2\nu^*)}{2\pi(1-\nu^*)}\, t_x t_y \ , \qquad c_{11} = -\frac{(5-2\nu^*)}{2\pi(1-\nu^*)}\,(t_x r_{0y} + t_y r_{0x}) \ ,$$
$$c_{12} = -\frac{(5-2\nu^*)}{2\pi(1-\nu^*)}\, r_{0x} r_{0y} \ , \qquad c_{13} = \frac{2}{\pi(1-\nu^*)}\, t_x t_y^3 \ , \qquad c_{14} = \frac{2}{\pi(1-\nu^*)}\, t_y^2 (3 t_x r_{0y} + t_y r_{0x}) \ ,$$
$$c_{15} = \frac{6}{\pi(1-\nu^*)}\, t_y (t_x r_{0y} + t_y r_{0x}) r_{0y} \ , \qquad c_{16} = \frac{2}{\pi(1-\nu^*)}\,(t_x r_{0y} + 3 t_y r_{0x}) r_{0y}^2 \ ,$$
$$c_{17} = \frac{2}{\pi(1-\nu^*)}\, r_{0x} r_{0y}^3 \ , \qquad c_{18} = -\frac{(1-2\nu^*)}{2\pi(1-\nu^*)}\, t_x t_y \ , \qquad c_{19} = -\frac{(1-2\nu^*)}{2\pi(1-\nu^*)}\,(t_x r_{0y} + t_y r_{0x}) \ ,$$
$$c_{20} = -\frac{(1-2\nu^*)}{2\pi(1-\nu^*)}\, r_{0x} r_{0y} \ , \qquad c_{21} = -\frac{1}{\pi(1-\nu^*)}\, t_x t_y (t_x^2 - t_y^2) \ ,$$
$$c_{22} = -\frac{1}{\pi(1-\nu^*)}\,[\,t_x^2 (t_x r_{0y} + 3 t_y r_{0x}) - t_y^2 (3 t_x r_{0y} + t_y r_{0x})\,] \ ,$$
$$c_{23} = -\frac{3}{\pi(1-\nu^*)}\,(t_x r_{0y} + t_y r_{0x})(t_x r_{0x} - t_y r_{0y}) \ ,$$
$$c_{24} = -\frac{1}{\pi(1-\nu^*)}\,[\,(3 t_x r_{0y} + t_y r_{0x}) r_{0x}^2 - (t_x r_{0y} + 3 t_y r_{0x}) r_{0y}^2\,] \ ,$$
$$c_{25} = -\frac{1}{\pi(1-\nu^*)}\,(r_{0x}^2 - r_{0y}^2) r_{0x} r_{0y} \ , \qquad c_{26} = \frac{1}{2\pi} \ ,$$
$$c_{27} = -\frac{1}{4\pi(1-\nu^*)}\,[\,(7-4\nu^*) t_y^2 - t_x^2\,] \ , \qquad c_{28} = -\frac{1}{2\pi(1-\nu^*)}\,[\,(7-4\nu^*) t_y r_{0y} - t_x r_{0x}\,] \ ,$$
$$c_{29} = -\frac{1}{4\pi(1-\nu^*)}\,[\,(7-4\nu^*) r_{0y}^2 - r_{0x}^2\,] \ , \qquad c_{30} = -\frac{1}{\pi(1-\nu^*)}\, t_y^2 (t_x^2 - t_y^2) \ ,$$
$$c_{31} = -\frac{2}{\pi(1-\nu^*)}\, t_y [\,t_x (t_x r_{0y} + t_y r_{0x}) - 2 t_y^2 r_{0y}\,] \ ,$$
$$c_{32} = -\frac{1}{\pi(1-\nu^*)}\,[\,(t_x^2 - 6 t_y^2) r_{0y}^2 + 4 t_x t_y r_{0x} r_{0y} + t_y^2 r_{0x}^2\,] \ ,$$
$$c_{33} = -\frac{2}{\pi(1-\nu^*)}\,[\,(t_x r_{0y} + t_y r_{0x}) r_{0x} - 2 t_y r_{0y}^2\,] r_{0y} \ , \qquad c_{34} = -\frac{1}{\pi(1-\nu^*)}\,(r_{0x}^2 - r_{0y}^2) r_{0y}^2 \ ,$$

$$c_{35} = -\frac{(1-2\nu^*)}{4\pi(1-\nu^*)} \ , \qquad c_{36} = \frac{1}{2\pi(1-\nu^*)}\left[\,(1-2\nu^*)t_x^2 + t_y^2\,\right] ,$$
$$c_{37} = \frac{1}{\pi(1-\nu^*)}\left[\,(1-2\nu^*)t_x r_{0x} + t_y r_{0y}\,\right] , \qquad c_{38} = \frac{1}{2\pi(1-\nu^*)}\left[\,(1-2\nu^*)r_{0x}^2 + r_{0y}^2\,\right] ,$$
$$c_{39} = \frac{(3-2\nu^*)}{2\pi(1-\nu^*)}\, t_x t_y \ , \qquad c_{40} = \frac{(3-2\nu^*)}{2\pi(1-\nu^*)}\left(t_x r_{0y} + t_y r_{0x}\right) , \qquad c_{41} = \frac{(3-2\nu^*)}{2\pi(1-\nu^*)}\, r_{0x} r_{0y} \ ,$$
$$c_{42} = -\frac{2}{\pi(1-\nu^*)}\, t_x^3 t_y \ , \qquad c_{43} = -\frac{2}{\pi(1-\nu^*)}\, t_x^2\left(t_x r_{0y} + 3t_y r_{0x}\right) ,$$
$$c_{44} = -\frac{6}{\pi(1-\nu^*)}\, t_x\left(t_x r_{0y} + t_y r_{0x}\right) r_{0x} \ , \qquad c_{45} = -\frac{2}{\pi(1-\nu^*)}\left(3t_x r_{0y} + t_y r_{0x}\right) r_{0x}^2 \ ,$$
$$c_{46} = -\frac{2}{\pi(1-\nu^*)}\, r_{0x}^3 r_{0y} \ , \qquad c_{47} = \frac{1}{2\pi(1-\nu^*)}\left[\,(1-2\nu^*)t_y^2 + t_x^2\,\right] ,$$
$$c_{48} = \frac{1}{\pi(1-\nu^*)}\left[\,(1-2\nu^*)t_y r_{0y} + t_x r_{0x}\,\right] , \qquad c_{49} = \frac{1}{2\pi(1-\nu^*)}\left[\,(1-2\nu^*)r_{0y}^2 + r_{0x}^2\,\right] ,$$
$$c_{50} = -\frac{1}{4\pi(1-\nu^*)}\left[\,(7-4\nu^*)t_x^2 - t_y^2\,\right] , \qquad c_{51} = -\frac{1}{2\pi(1-\nu^*)}\left[\,(7-4\nu^*)t_x r_{0x} - t_y r_{0y}\,\right] ,$$
$$c_{52} = -\frac{1}{4\pi(1-\nu^*)}\left[\,(7-4\nu^*)r_{0x}^2 - r_{0y}^2\,\right] , \qquad c_{53} = -\frac{1}{\pi(1-\nu^*)}\, t_x^2\left(t_y^2 - t_x^2\right) ,$$
$$c_{54} = -\frac{2}{\pi(1-\nu^*)}\, t_x\left[\, t_y\left(t_x r_{0y} + t_y r_{0x}\right) - 2t_x^2 r_{0x}\,\right] ,$$
$$c_{55} = -\frac{1}{\pi(1-\nu^*)}\left[\, t_x^2 r_{0y}^2 + 4t_x t_y r_{0x} r_{0y} + \left(t_y^2 - 6t_x^2\right) r_{0x}^2\,\right] ,$$
$$c_{56} = -\frac{2}{\pi(1-\nu^*)}\left[\,\left(t_x r_{0y} + t_y r_{0x}\right) r_{0y} - 2t_x r_{0x}^2\,\right] r_{0x} \ , \qquad c_{57} = -\frac{1}{\pi(1-\nu^*)}\left(r_{0y}^2 - r_{0x}^2\right) r_{0x}^2 \ ,$$
$$c_{58} = -\frac{1}{2\pi(1-\nu^*)}\left[\,(3-2\nu^*)t_y^2 + t_x^2\,\right] , \qquad c_{59} = -\frac{1}{\pi(1-\nu^*)}\left[\,(3-2\nu^*)t_y r_{0y} + t_x r_{0x}\,\right] ,$$
$$c_{60} = -\frac{1}{2\pi(1-\nu^*)}\left[\,(3-2\nu^*)r_{0y}^2 + r_{0x}^2\,\right] .$$

12. Values of the Constants d_i

The values of the constants d_i appearing in eqns. (51) through (54) are listed below. Note that the values of the constants c_i are listed directly above in Sec. 11.

$d_1 = n_x c_1$, $\quad d_2 = n_x c_2 + n_y c_{18}$, $\quad d_3 = n_x c_3 + n_y c_{19}$,
$d_4 = n_x c_4 + n_y c_{20}$, $\quad d_5 = n_x c_5 + n_y c_{21}$, $\quad d_6 = n_x c_6 + n_y c_{22}$,
$d_7 = n_x c_7 + n_y c_{23}$, $\quad d_8 = n_x c_8 + n_y c_{24}$, $\quad d_9 = n_x c_9 + n_y c_{25}$,
$d_{10} = n_y c_{26}$, $\quad d_{11} = n_x c_{10} + n_y c_{27}$, $\quad d_{12} = n_x c_{11} + n_y c_{28}$,
$d_{13} = n_x c_{12} + n_y c_{29}$, $\quad d_{14} = n_x c_{13} + n_y c_{30}$, $\quad d_{15} = n_x c_{14} + n_y c_{31}$,
$d_{16} = n_x c_{15} + n_y c_{32}$, $\quad d_{17} = n_x c_{16} + n_y c_{33}$, $\quad d_{18} = n_x c_{17} + n_y c_{34}$,
$d_{19} = n_y c_{35}$, $\quad d_{20} = n_x c_{18} + n_y c_{36}$, $\quad d_{21} = n_x c_{19} + n_y c_{37}$,
$d_{22} = n_x c_{20} + n_y c_{38}$, $\quad d_{23} = n_x c_{21} - n_y c_5$, $\quad d_{24} = n_x c_{22} - n_y c_6$,
$d_{25} = n_x c_{23} - n_y c_7$, $\quad d_{26} = n_x c_{24} - n_y c_8$, $\quad d_{27} = n_x c_{25} - n_y c_9$,
$d_{28} = n_x c_{26}$, $\quad d_{29} = n_x c_{27} + n_y c_{39}$, $\quad d_{30} = n_x c_{28} + n_y c_{40}$,
$d_{31} = n_x c_{29} + n_y c_{41}$, $\quad d_{32} = n_x c_{30} - n_y c_{13}$, $\quad d_{33} = n_x c_{31} - n_y c_{14}$,
$d_{34} = n_x c_{32} - n_y c_{15}$, $\quad d_{35} = n_x c_{33} - n_y c_{16}$, $\quad d_{36} = n_x c_{34} - n_y c_{17}$,
$d_{37} = n_y c_{26}$, $\quad d_{38} = n_x c_{39} + n_y c_{50}$, $\quad d_{39} = n_x c_{40} + n_y c_{51}$,
$d_{40} = n_x c_{41} + n_y c_{52}$, $\quad d_{41} = n_x c_{42} + n_y c_{53}$, $\quad d_{42} = n_x c_{43} + n_y c_{54}$,

$d_{43} = n_x c_{44} + n_y c_{55}$, $d_{44} = n_x c_{45} + n_y c_{56}$, $d_{45} = n_x c_{46} + n_y c_{57}$,
$d_{46} = n_x c_{35}$, $d_{47} = n_x c_{47} + n_y c_{18}$, $d_{48} = n_x c_{48} + n_y c_{19}$,
$d_{49} = n_x c_{49} + n_y c_{20}$, $d_{50} = -n_x c_5 - n_y c_{21}$, $d_{51} = -n_x c_6 - n_y c_{22}$,
$d_{52} = -n_x c_7 - n_y c_{23}$, $d_{53} = -n_x c_8 - n_y c_{24}$, $d_{54} = -n_x c_9 - n_y c_{25}$,
$d_{55} = n_x c_{26}$, $d_{56} = n_x c_{50} + n_y c_{10}$, $d_{57} = n_x c_{51} + n_y c_{11}$,
$d_{58} = n_x c_{52} + n_y c_{12}$, $d_{59} = n_x c_{53} - n_y c_{42}$, $d_{60} = n_x c_{54} - n_y c_{43}$,
$d_{61} = n_x c_{55} - n_y c_{44}$, $d_{62} = n_x c_{56} - n_y c_{45}$, $d_{63} = n_x c_{57} - n_y c_{46}$,
$d_{64} = n_y c_1$, $d_{65} = n_x c_{18} + n_y c_{58}$, $d_{66} = n_x c_{19} + n_y c_{59}$,
$d_{67} = n_x c_{20} + n_y c_{60}$, $d_{68} = -n_x c_{21} + n_y c_5$, $d_{69} = -n_x c_{22} + n_y c_6$,
$d_{70} = -n_x c_{23} + n_y c_7$, $d_{71} = -n_x c_{24} + n_y c_8$, $d_{72} = -n_x c_{25} + n_y c_9$,

$$d_{73} = \frac{(3-4\nu^*)S}{4\pi(1-\nu^*)}\, t_x \ , \qquad d_{74} = \frac{(3-4\nu^*)S}{4\pi(1-\nu^*)}\, r_{0x} \ , \qquad d_{75} = -\frac{S}{2\pi(1-\nu^*)}\, t_x t_y^2 \ ,$$

$$d_{76} = -\frac{S}{2\pi(1-\nu^*)}\, t_y\big(2t_x r_{0y} + t_y r_{0x}\big) \ , \qquad d_{77} = -\frac{S}{2\pi(1-\nu^*)}\, \big(t_x r_{0y} + 2t_y r_{0x}\big) r_{0y} \ ,$$

$$d_{78} = -\frac{S}{2\pi(1-\nu^*)}\, r_{0x} r_{0y}^2 \ , \qquad d_{79} = \frac{(5-4\nu^*)S}{4\pi(1-\nu^*)}\, t_y \ , \qquad d_{80} = \frac{(5-4\nu^*)S}{4\pi(1-\nu^*)}\, r_{0y} \ ,$$

$$b_{81} = -\frac{S}{2\pi(1-\nu^*)}\, t_y^3 \ , \qquad d_{82} = -\frac{3S}{2\pi(1-\nu^*)}\, t_y^2 r_{0y} \ , \qquad d_{83} = -\frac{3S}{2\pi(1-\nu^*)}\, t_y r_{0y}^2 \ ,$$

$$d_{84} = -\frac{S}{2\pi(1-\nu^*)}\, r_{0y}^3 \ , \qquad d_{85} = -\frac{S}{4\pi(1-\nu^*)}\, t_y \ , \qquad d_{86} = -\frac{S}{4\pi(1-\nu^*)}\, r_{0y} \ ,$$

$$d_{87} = \frac{S}{2\pi(1-\nu^*)}\, t_x^2 t_y \ , \qquad d_{88} = \frac{S}{2\pi(1-\nu^*)}\, t_x\big(t_x r_{0y} + 2t_y r_{0x}\big) \ ,$$

$$d_{89} = \frac{S}{2\pi(1-\nu^*)}\, \big(2t_x r_{0y} + t_y r_{0x}\big) r_{0x} \ , \qquad d_{90} = \frac{S}{2\pi(1-\nu^*)}\, r_{0x}^2 r_{0y} \ , \qquad d_{91} = -\frac{S}{4\pi(1-\nu^*)}\, t_x \ ,$$

$$d_{92} = -\frac{S}{4\pi(1-\nu^*)}\, r_{0x} \ , \qquad d_{93} = \frac{(5-4\nu^*)S}{4\pi(1-\nu^*)}\, t_x \ , \qquad d_{94} = \frac{(5-4\nu^*)S}{4\pi(1-\nu^*)}\, r_{0x} \ ,$$

$$d_{95} = -\frac{S}{2\pi(1-\nu^*)}\, t_x^3 \ , \qquad d_{96} = -\frac{3S}{2\pi(1-\nu^*)}\, t_x^2 r_{0x} \ , \qquad d_{97} = -\frac{3S}{2\pi(1-\nu^*)}\, t_x r_{0x}^2 \ ,$$

$$d_{98} = -\frac{S}{2\pi(1-\nu^*)}\, r_{0x}^3 \ , \qquad d_{99} = \frac{(3-4\nu^*)S}{4\pi(1-\nu^*)}\, t_y \ , \qquad d_{100} = \frac{(3-4\nu^*)S}{4\pi(1-\nu^*)}\, r_{0y} \ .$$

13. List of Indefinite Integrals

The definite integrals I_1 through I_{17} appearing above in eqns. (43), (44), (53) and (54) may be evaluated with the indefinite integrals listed in this section. The quantities Q, β and r_0^2 occurring in the expressions below were defined above in eqns. (39). Additionally,

$$\gamma = \sqrt{r_0^2 - \beta^2} \ , \qquad \delta = \tan^{-1}\left[\frac{\gamma}{t+\beta}\right] \ .$$

13.1. Non-Degenerate Case $(\gamma \neq 0)$

Since Q has two complex conjugate roots, *i.e.,* $Q = [\, t - (-\beta + i\gamma)\,][\, t - (-\beta - i\gamma)\,]$, partial fractions (with complex variables) may be used to calculate the integrals $I_4(t)$, $I_9(t)$, $I_{11}(t)$ and $I_{17}(t)$. Integration by parts may then be used to calculate the remaining integrals.

$$I_1(t) = \int \frac{t^3\mathrm{d}t}{Q} = -\frac{1}{2}(r_0^2 - 4\beta^2)\ln Q - 2\beta\gamma\delta - \beta(r_0^2 - 2\beta^2)\frac{\delta}{\gamma} + \frac{1}{2}t^2 - 2\beta t$$

$$I_2(t) = \int \frac{t^2 \mathrm{d}t}{Q} = -\beta \ln Q + (r_0^2 - 2\beta^2)\frac{\delta}{\gamma} + t$$

$$I_3(t) = \int \frac{t \mathrm{d}t}{Q} = \frac{1}{2}\ln Q + \beta \frac{\delta}{\gamma}$$

$$I_4(t) = \int \frac{\mathrm{d}t}{Q} = -\frac{\delta}{\gamma}$$

$$I_5(t) = \int \frac{t^4 \mathrm{d}t}{Q^2} = \frac{1}{2\gamma^2}\left[-4\beta\gamma^2 \ln Q + \frac{t^5 + \beta t^4}{Q} + (3r_0^2 - 8\beta^2)\gamma\delta - r_0^2\beta^2\frac{\delta}{\gamma} - t^3 + \beta t^2 + (3r_0^2 - 4\beta^2)t\right]$$

$$I_6(t) = \int \frac{t^3 \mathrm{d}t}{Q^2} = \frac{1}{2\gamma^2}\left[\gamma^2 \ln Q + \frac{t^4 + \beta t^3}{Q} + 2\beta\gamma\delta + r_0^2\beta\frac{\delta}{\gamma} - t^2 + \beta t\right]$$

$$I_7(t) = \int \frac{t^2 \mathrm{d}t}{Q^2} = \frac{1}{2\gamma^2}\left[\frac{t^3 + \beta t^2}{Q} - r_0^2\frac{\delta}{\gamma} - t\right]$$

$$I_8(t) = \int \frac{t \mathrm{d}t}{Q^2} = \frac{1}{2\gamma^2}\left[\frac{t^2 + \beta t}{Q} + \beta\frac{\delta}{\gamma}\right]$$

$$I_9(t) = \int \frac{\mathrm{d}t}{Q^2} = \frac{1}{2\gamma^2}\left[\frac{t + \beta}{Q} - \frac{\delta}{\gamma}\right]$$

$$I_{10}(t) = \int t \ln Q \, \mathrm{d}t = \frac{1}{2}(t^2 + r_0^2 - 2\beta^2)\ln Q + 2\beta\gamma\delta - \frac{1}{2}t^2 + \beta t$$

$$I_{11}(t) = \int \ln Q \, \mathrm{d}t = (t + \beta)\ln Q - 2\gamma\delta - 2t$$

$$I_{12}(t) = \int \frac{t^5 \mathrm{d}t}{Q^3} = \frac{1}{8\gamma^4}\left[4\gamma^4 \ln Q + \frac{2\gamma^2(t^6 + \beta t^5)}{Q^2} + \frac{-3t^6 - 9\beta t^5 - (r_0^2 + 5\beta^2)t^4}{Q} + \beta(17r_0^2 - 14\beta^2)\gamma\delta - \beta(2r_0^4 - 11r_0^2\beta^2 + 6\beta^4)\frac{\delta}{\gamma} + 3t^4 + 3\beta t^3 - (4r_0^2 - \beta^2)t^2 + \beta(7r_0^2 - 4\beta^2)t\right]$$

$$I_{13}(t) = \int \frac{t^4 \mathrm{d}t}{Q^3} = \frac{1}{8\gamma^4}\left[\frac{2\gamma^2(t^5 + \beta t^4)}{Q^2} + \frac{-2t^5 - 7\beta t^4 - (r_0^2 + 4\beta^2)t^3}{Q} - 4\gamma^3\delta + (r_0^4 - 8r_0^2\beta^2 + 4\beta^4)\frac{\delta}{\gamma} + 2t^3 + 3\beta t^2 - 3r_0^2 t\right]$$

$$I_{14}(t) = \int \frac{t^3 \mathrm{d}t}{Q^3} = \frac{1}{8\gamma^4}\left[\frac{2\gamma^2(t^4 + \beta t^3)}{Q^2} + \frac{-t^4 - 5\beta t^3 - (r_0^2 + 3\beta^2)t^2}{Q} - 2\beta\gamma\delta + \beta(5r_0^2 - 2\beta^2)\frac{\delta}{\gamma} + t^2 + 3\beta t\right]$$

$$I_{15}(t) = \int \frac{t^2 \mathrm{d}t}{Q^3} = \frac{1}{8\gamma^4}\left[\frac{2\gamma^2(t^3 + \beta t^2)}{Q^2} + \frac{-3\beta t^2 - (r_0^2 + 2\beta^2)t}{Q} - (r_0^2 + 2\beta^2)\frac{\delta}{\gamma}\right]$$

$$I_{16}(t) = \int \frac{t \mathrm{d}t}{Q^3} = \frac{1}{8\gamma^4}\left[\frac{2\gamma^2(t^2 + \beta t)}{Q^2} + \frac{2t^2 + \beta t - \beta^2}{Q} + 3\beta\frac{\delta}{\gamma}\right]$$

$$I_{17}(t) = \int \frac{\mathrm{d}t}{Q^3} = \frac{1}{8\gamma^4}\left[\frac{2\gamma^2(t + \beta)}{Q^2} + \frac{3(t + \beta)}{Q} - 3\frac{\delta}{\gamma}\right]$$

13.2. Degenerate Case ($\gamma = 0$)

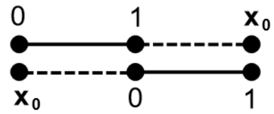

Figure 10. Case where $\gamma = 0$.

When the singularity $\mathbf{x_0}$ is on the line containing the line segment of the boundary element, as pictured at left in Fig. 10, then $\gamma = 0$, or $r_0^2 = \beta^2$, and $Q = (t+\beta)^2$. In this case one may use

$$\lim_{\gamma \to 0} \frac{\delta}{\gamma} = \frac{1}{t+\beta}$$

in the expressions for $I_1(t)$ through $I_4(t)$. For the integrals $I_5(t)$ through $I_9(t)$ and $I_{12}(t)$ through $I_{17}(t)$, the following expressions must be used, which expressions are obtained easily by using the variable substitution $w = t + \beta$.

$$I_5(t) = \int \frac{t^4 \mathrm{d}t}{Q^2} = (t+\beta) - 4\beta \ln|t+\beta| - \frac{6\beta^2}{(t+\beta)} + \frac{2\beta^3}{(t+\beta)^2} - \frac{\beta^4}{3(t+\beta)^3}$$

$$I_6(t) = \int \frac{t^3 \mathrm{d}t}{Q^2} = \ln|t+\beta| + \frac{3\beta}{(t+\beta)} - \frac{3\beta^2}{2(t+\beta)^2} + \frac{\beta^3}{3(t+\beta)^3}$$

$$I_7(t) = \int \frac{t^2 \mathrm{d}t}{Q^2} = -\frac{1}{(t+\beta)} + \frac{\beta}{(t+\beta)^2} - \frac{\beta^2}{3(t+\beta)^3}$$

$$I_8(t) = \int \frac{t \mathrm{d}t}{Q^2} = -\frac{1}{2(t+\beta)^2} + \frac{\beta}{3(t+\beta)^3}$$

$$I_9(t) = \int \frac{\mathrm{d}t}{Q^2} = -\frac{1}{3(t+\beta)^3}$$

$$I_{12}(t) = \int \frac{t^5 \mathrm{d}t}{Q^3} = \ln|t+\beta| + \frac{5\beta}{(t+\beta)} - \frac{5\beta^2}{(t+\beta)^2} + \frac{10\beta^3}{3(t+\beta)^3} - \frac{5\beta^4}{4(t+\beta)^4} + \frac{\beta^5}{5(t+\beta)^5}$$

$$I_{13}(t) = \int \frac{t^4 \mathrm{d}t}{Q^3} = -\frac{1}{(t+\beta)} + \frac{2\beta}{(t+\beta)^2} - \frac{2\beta^2}{(t+\beta)^3} + \frac{\beta^3}{(t+\beta)^4} - \frac{\beta^4}{5(t+\beta)^5}$$

$$I_{14}(t) = \int \frac{t^3 \mathrm{d}t}{Q^3} = -\frac{1}{2(t+\beta)^2} + \frac{\beta}{(t+\beta)^3} - \frac{3\beta^2}{4(t+\beta)^4} + \frac{\beta^3}{5(t+\beta)^5}$$

$$I_{15}(t) = \int \frac{t^2 \mathrm{d}t}{Q^3} = -\frac{1}{3(t+\beta)^3} + \frac{\beta}{2(t+\beta)^4} - \frac{\beta^2}{5(t+\beta)^5}$$

$$I_{16}(t) = \int \frac{t \mathrm{d}t}{Q^3} = -\frac{1}{4(t+\beta)^4} + \frac{\beta}{5(t+\beta)^5}$$

$$I_{17}(t) = \int \frac{\mathrm{d}t}{Q^3} = -\frac{1}{5(t+\beta)^5}$$

14. Analytical Example in Cartesian Coordinates

Consider the stress field

$$\sigma_{xx} = \frac{12V}{H^3}(L-x)y\,, \qquad \sigma_{yy} = 0\,, \qquad \sigma_{xy} = \frac{3V}{2H^3}(4y^2 - H^2)\,, \tag{69}$$

which spans the rectangular domain (a cantilever beam) of Fig. 11 below. Note that

$$\int_{-H/2}^{H/2} \sigma_{xy}(L,y)\mathrm{d}y = -V\,, \qquad \int_{-H/2}^{H/2} y\sigma_{xx}(0,y)\mathrm{d}y = VL\,, \tag{70}$$

so that $V > 0$ is the net shear force applied to the ends of the beam, and a moment VL is applied to the left end of the beam. Also, eqns. (69) satisfy equilibrium $\sigma_{ij,i} = 0$ identically. With eqns. (69) and Hooke's Law (1), the strain components $\varepsilon_{ij} = 1/2\ (u_{i,j} + u_{j,i})$ are

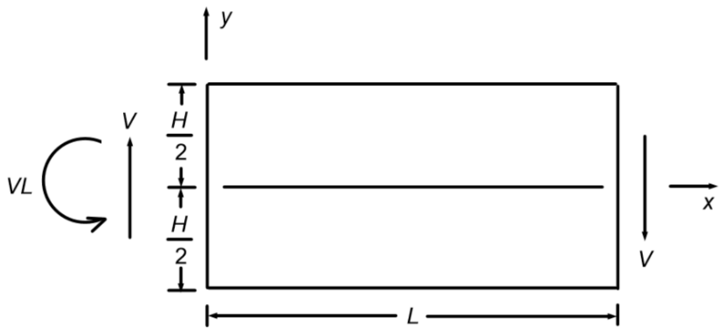

Figure 11. Domain of a cantilever beam.

$$\varepsilon_{xx} = \frac{12VS(1-\nu^*)}{H^3}\,(L-x)y\,,$$
$$\varepsilon_{yy} = -\frac{12VS\nu^*}{H^3}\,(L-x)y\,, \qquad (71)$$
$$\varepsilon_{xy} = \frac{3VS}{2H^2}\,(4y^2 - H^2)\,.$$

Finally, integrating the strains (71) such that $u_x(0,0) = u_y(0,0) = 0$ and $u_x(0, H/2) = 0$, one obtains the displacement field

$$u_x = \frac{VS}{2H^3}\,[\,12(1-\nu^*)(2Lx - x^2)y + 4(2-\nu^*)y^3 - (2-\nu^*)H^2 y\,]\,,$$
$$u_y = -\frac{VS}{2H^3}\,[\,12\nu^*(L-x)y^2 + 4(1-\nu^*)(3Lx^2 - x^3) + (4+\nu^*)H^2 x\,]\,. \qquad (72)$$

Consistent boundary conditions are then

$$\text{on } x = 0\,, \quad T_x = -\frac{12VL}{H^3}\,y \,, \quad T_y = -\frac{3V}{2H^3}\,(4y^2 - H^2) \qquad \text{on } x = L\,, \quad T_x = 0\,, \quad T_y = \frac{3V}{2H^3}\,(4y^2 - H^2)$$
$$\text{and on } y = \pm\frac{H}{2}\,, \quad T_x = 0\,, \quad T_y = 0 \qquad (73)$$

along with the conditions $u_x(0,0) = u_y(0,0) = 0$ and $u_x(0, H/2) = 0$.

15. Analytical Example in Polar Coordinates

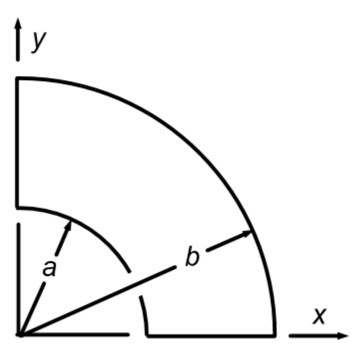

Figure 12. Quarter-annular domain.

In polar coordinates, the quarter-annular domain shown at left in Fig. 12 is subjected to the boundary conditions

$$u_r(a,\theta) = u_\theta(a,\theta) = 0\,,$$
$$T_r(b,\theta) = \frac{2F}{b}\cos 4\theta\,, \qquad T_\theta(b,\theta) = 0\,,$$
$$T_r(r,0) = 0\,, \qquad u_\theta(r,0) = 0\,, \qquad (74)$$
$$T_r(r,\pi/2) = 0\,, \qquad u_\theta(r,\pi/2) = 0\,,$$

where

$$F = \int_{0}^{\pi/8} T_r(b,\theta)\, b\mathrm{d}\theta \,. \tag{75}$$

The boundary conditions (74) may be satisfied with a displacement field of the form

$$u_r = f(r)\cos 4\theta \;, \qquad u_\theta = g(r)\sin 4\theta \,. \tag{76}$$

Substituting eqns. (76) into the strain-displacement relations (58), one obtains the strains

$$\varepsilon_{rr} = f'\cos 4\theta \;, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}(\,f + 4g)\cos 4\theta \;, \qquad \varepsilon_{r\theta} = \frac{1}{2}\Big(\,g' - \frac{1}{r}g - \frac{4}{r}f\,\Big)\sin 4\theta \;, \tag{77}$$

and via Hooke's Law (4), eqns. (77) give the stresses

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{S(1-2\nu^*)}\left[\,(1-\nu^*)f' + \frac{\nu^*}{r}f + \frac{4\nu^*}{r}g\,\right]\cos 4\theta \,, \\
\sigma_{\theta\theta} &= \frac{1}{S(1-2\nu^*)}\left[\,\nu^* f' + \frac{(1-\nu^*)}{r}f + \frac{4(1-\nu^*)}{r}g\,\right]\cos 4\theta \,, \\
\sigma_{r\theta} &= \frac{1}{2S}\left(\,g' - \frac{1}{r}g - \frac{4}{r}f\,\right)\sin 4\theta \,.
\end{aligned} \tag{78}$$

Note that eqns. (76) and (78) satisfy the boundary conditions at $\theta = 0$ and $\theta = \pi/2$ identically. Notwithstanding, in polar coordinates, the equilibrium equations are

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\theta,\theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \,, \qquad \sigma_{r\theta,r} + \frac{1}{r}\sigma_{\theta\theta,\theta} + \frac{2}{r}\sigma_{r\theta} = 0 \,. \tag{79}$$

Now, substitution of the stresses (78) into the equilibrium eqns. (79) yields the coupled pair of ordinary differential equations

$$\begin{aligned}
&(1-\nu^*)f'' + \frac{(1-\nu^*)}{r}f' - \frac{(9-17\nu^*)}{r^2}f + \frac{2}{r}g' - \frac{2(3-4\nu^*)}{r^2}g = 0 \,, \\
&(1-2\nu^*)g'' + \frac{(1-2\nu^*)}{r}g' - \frac{(33-34\nu^*)}{r^2}g - \frac{4}{r}f' - \frac{4(3-4\nu^*)}{r^2}f = 0 \,.
\end{aligned} \tag{80}$$

By assuming functions of the form

$$f = kr^p \,, \qquad g = lr^p \,, \tag{81}$$

eqns. (80) become

$$\begin{bmatrix} [\,(1-\nu^*)p^2 - (9-17\nu^*)\,] & 2[\,p - (3-4\nu^*)\,] \\ -4[\,p + (3-4\nu^*)\,] & [\,(1-2\nu^*)p - (33-34\nu^*)\,] \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \tag{82}$$

which has nontrivial solutions if the determinant of coefficients is zero, *viz.,*

$$p^4 - 34p^2 + 225 = 0 \,. \tag{83}$$

Thus,

$$p = -5\,, -3\,, 3\,, 5\,. \tag{84}$$

Next, using the four null vectors of eqns. (82) as generated by the powers (84), one obtains the relations between the eight constants k_i and l_i

$$p = -5 \;\Rightarrow\; l_1 = k_1\,, \qquad p = -3 \;\Rightarrow\; l_2 = \frac{2\nu^*}{3-2\nu^*}k_2\,,$$

$$p = 3 \;\Rightarrow\; l_3 = -k_3\,, \qquad p = 5 \;\Rightarrow\; l_4 = -\frac{2(2-\nu^*)}{1+2\nu^*}k_4\,. \tag{85}$$

Finally, the functions f and g are then

$$f = \frac{k_1}{r^5} + \frac{k_2}{r^3} + k_3 r^3 + k_4 r^5\,, \qquad g = \frac{l_1}{r^5} + \frac{l_2}{r^3} + l_3 r^3 + l_4 r^5\,. \tag{86}$$

Turning attention to the boundary conditions at $r = a$ and $r = b$, eqns. (76), (78), (85) and (86) give the system

$$\begin{bmatrix} 1/a^5 & 1/a^3 & a^3 & a^5 \\ 1/a^5 & 2\nu^*/[\,(3-2\nu^*)a^3\,] & -a^3 & -2(2-\nu^*)a^5/(1+2\nu^*) \\ -5/b^5 & -9/[\,(3-2\nu^*)b^3\,] & 3b^3 & 5b^5/(1+2\nu^*) \\ -5/b^5 & -6/[\,(3-2\nu^*)b^3\,] & -3b^3 & -10b^5/(1+2\nu^*) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2FS \\ 0 \end{bmatrix} \tag{87}$$

to solve for the constants k_i. The first of eqns. (87) is from $u_r(a,\theta) = 0$; the second is from $u_\theta(a,\theta) = 0$; the third, from $T_r(b,\theta) = 2F\cos 4\theta/b$; and the fourth, $T_\theta(b,\theta) = 0$. Instead of solving eqns. (87) algebraically, they were solved numerically using the constants

$$E = 3\times10^7 \text{ psi}\,, \quad \nu = 0.3\,, \quad a = 36 \text{ in}\,, \quad b = 72 \text{ in}\,, \quad F = 10{,}000 \text{ lb} \tag{88}$$

for plane stress. The results are

$$\begin{aligned} k_1 &= 4.468\,854\,986\,101\,9630\times10^{3} & l_1 &= 4.468\,854\,986\,101\,9630\times10^{3} \\ k_2 &= -6.240\,733\,337\,857\,3010\times10^{0} & l_2 &= -1.134\,678\,788\,701\,3274\times10^{0} \\ k_3 &= 1.437\,736\,140\,089\,1819\times10^{-9} & l_3 &= -1.437\,736\,140\,089\,1819\times10^{-9} \\ k_4 &= -1.194\,905\,134\,758\,0960\times10^{-13} & l_4 &= 2.892\,928\,220\,993\,2855\times10^{-13} \end{aligned}\,, \tag{89}$$

which constants solve the problem at hand.

16. Numerical Example – Cartesian Coordinates

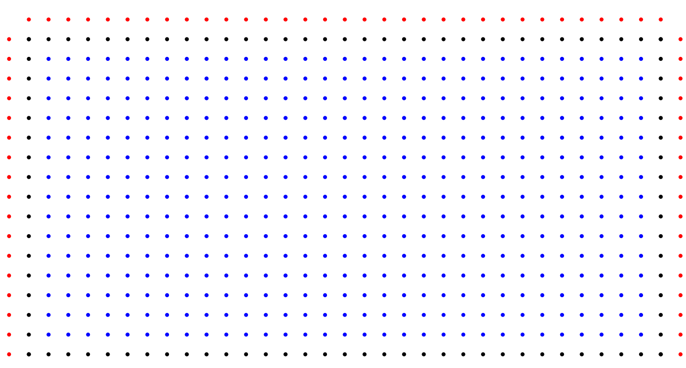

Figure 13. Computational grid used for the analysis as described in the text.

Here the problem solved earlier in Sec. 14 is solved numerically. The grid used for the calculations is shown at left in Fig. 13. The black and blue points in the figure consist of a 33 by 17 array of 561 points. The black points are on the boundary of the domain while the blue points are in the interior. Between the black points are the 96 boundary elements. Note that on the corners the nodes are double nodes, with one of them belonging to one face, and the other to the adjacent face. At these double nodes, the continuity of the displacements is not enforced explicitly during the boundary solution, but they come out to be continuous for all practical purposes. After the boundary solution though, the displacements at the corner double nodes are averaged. In any case, including the double nodes at the corners, there are 100 boundary nodes. The 100 red points in the figure are where the singularities are placed to generate the two required equations for each boundary node. Finally, the constants used in the analysis are

$$L = 10 \text{ in}\,, \qquad H = 5 \text{ in}\,, \qquad V = 10{,}000 \text{ lb}\,, \qquad E = 3 \times 10^7 \text{ psi}\,, \qquad \nu = 0.3\,, \qquad (90)$$

and plane stress is assumed.

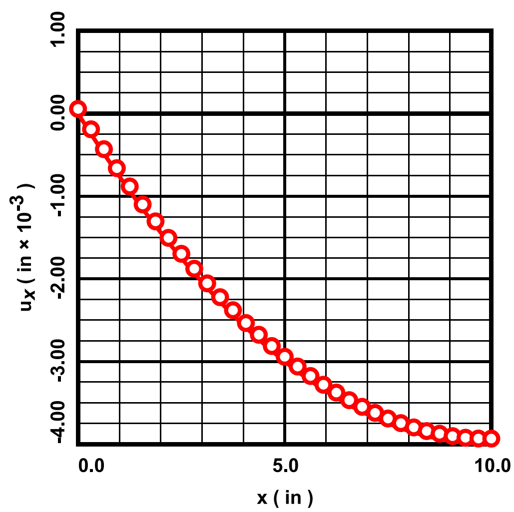

Figure 14. Analytical and numerical results for the displacement u_x along $y = -2.5$ in.

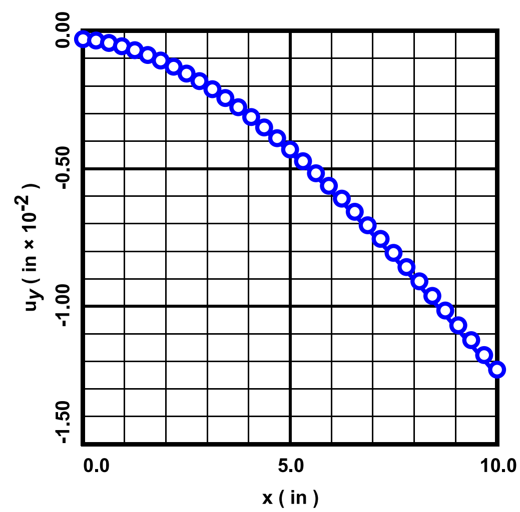

Figure 15. Analytical and numerical results for the displacement u_y along $y = -2.5$ in.

Figures 14 and 15 above show the exact (solid curves) and numerically calculated (plotted points) displacement components along the bottom face ($y = -2.5$ in) of the domain. The numerically calculated values are quite accurate, but they do under-estimate the exact values somewhat. This is due to the parabolic force distribution along the right boundary of the grid being represented by sixteen line segments, which slightly under-estimates the total applied shear force.

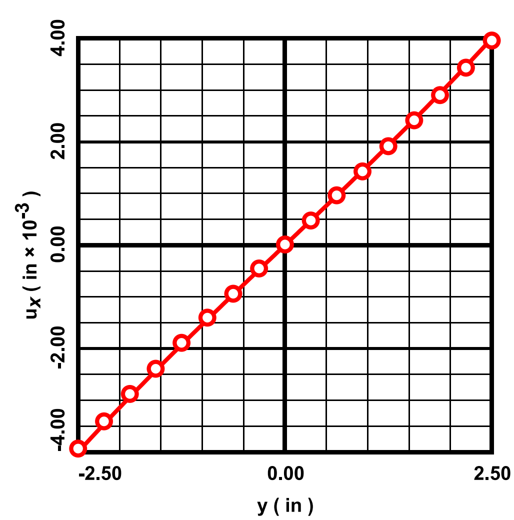

Figure 16. Analytical and numerical results for the displacement u_x along $x = 10$ in.

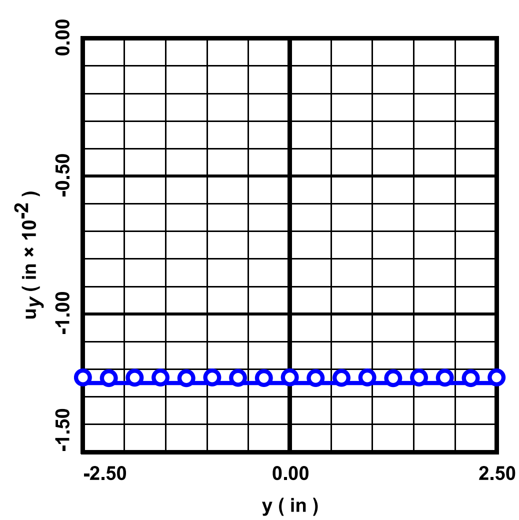

Figure 17. Analytical and numerical results for the displacement u_y along $x = 10$ in.

Figures 16 and 17 above show the results for the displacement components along the right edge $x = 10$ in of the domain. Once again, the numerically calculated values are quite accurate, although slightly under-estimated. Figure 18 below shows the results for the displacements along the top face $y = 2.5$ in of the domain. As is evident, the numerically calculated values are highly accurate. Figure 19 below shows the displacement components along the left face of the grid at $x = 0$ in. This is the least accurate part of the solution, especially for the component u_x. Also, note the slight oscillation exhibited by u_y near $y = 0$ in. Perhaps the behavior of the numerical results on the left face is due to the discrete displacement conditions $u_x(0,0) = u_y(0,0) = 0$ and $u_x(0, H/2) = 0$.

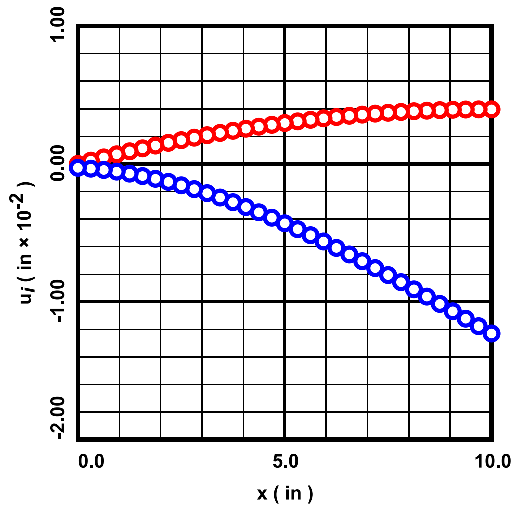

Figure 18. Analytical and numerical results for the displacements u_x (red) and u_y (blue) along $y = 2.5$ in.

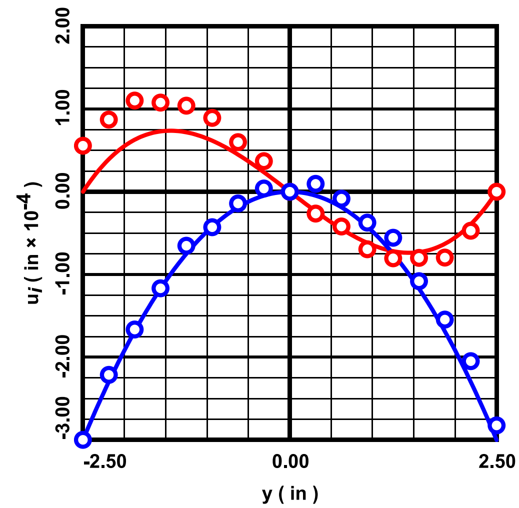

Figure 19. Analytical and numerical results for the displacements u_x (red) and u_y (blue) along $x = 0$ in.

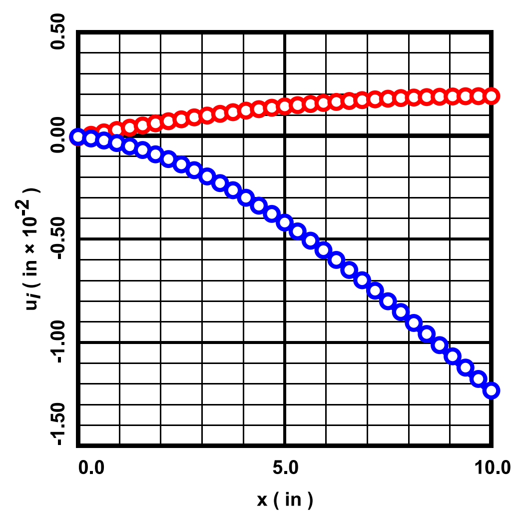

Figure 20. Analytical and numerical results for the displacements u_x (red) and u_y (blue) along $y = 1.25$ in.

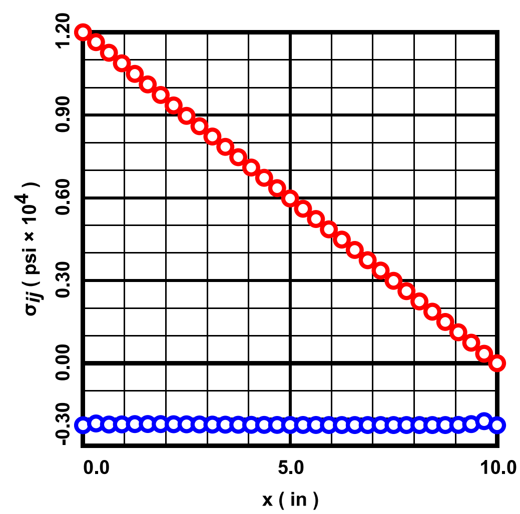

Figure 21. Analytical and numerical results for the stresses σ_{xx} (red) and σ_{yy} (blue) along $y = 1.25$ in.

Figures 20 and 21 above show the results for the displacement and stress components along the horizontal line $y = 1.25$ in in the domain. Except for the slight blips in σ_{yy} near the left and right boundaries, all the calculated results are highly accurate. Finally, Figs. 22 and 23 below show the results for the displacement and stress components along the vertical line $x = 7.5$ in in the domain. Again, note the slight under-estimation of the displacement u_y. The numerically calculated stresses, on the other hand, are highly accurate.

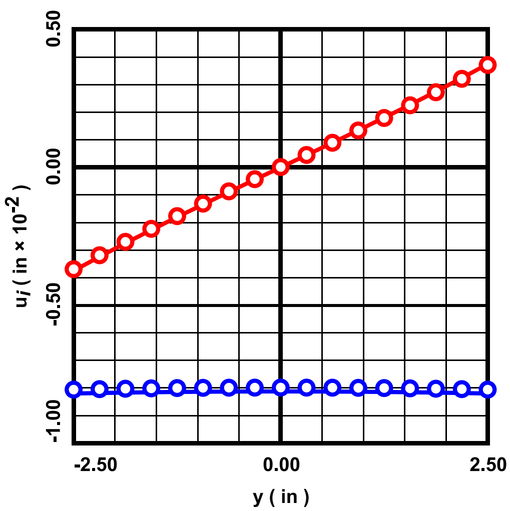

Figure 22. Analytical and numerical results for the displacements u_x (red) and u_y (blue) along $x = 7.5$ in.

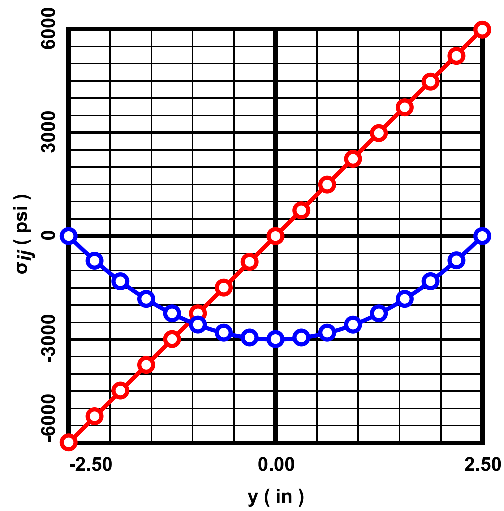

Figure 23. Analytical and numerical results for the stresses σ_{xx} (red) and σ_{yy} (blue) along $x = 7.5$ in.

17. Numerical Example – Polar Coordinates

Here the problem solved analytically in Sec. 15 is solved numerically. The grid used in the analysis is shown below in Fig. 24. The black and blue points in the figure consist of a 25 (radial) by 37 (tangential) array of 925 points. The black points are on the boundary, and between the black points are the 120 boundary elements. As was the case before, the black points on the corners are double nodes. During the boundary solution the continuity of the displacements at the corner double nodes is not explicitly enforced, but for all practical purposes, the corner displacements come out to be continuous. Nevertheless, after the boundary solution, these corner displacements are averaged. Including the double corner nodes, there are 124 boundary nodes, and the 124 red points in the figure are where the singularities are placed to generate the two equations needed for each boundary node. Finally, the constants used in the analysis are given above by eqns. (89).

Figures 25 and 26 below show the exact (solid curves) and numerically calculated (plotted points) of the boundary solution along $\theta = 0$. As Fig. 25 shows, the numerically calculated values of the displacement u_x are highly accurate. Figure 26 shows the results for the traction component T_y. While the magnitudes of T_y are not all that inaccurate, the numerical results possess slight oscillations on $r \in (36,64)$ in. While this is the most inaccurate part of the solution, the author is not entirely sure of why the oscillations occur.

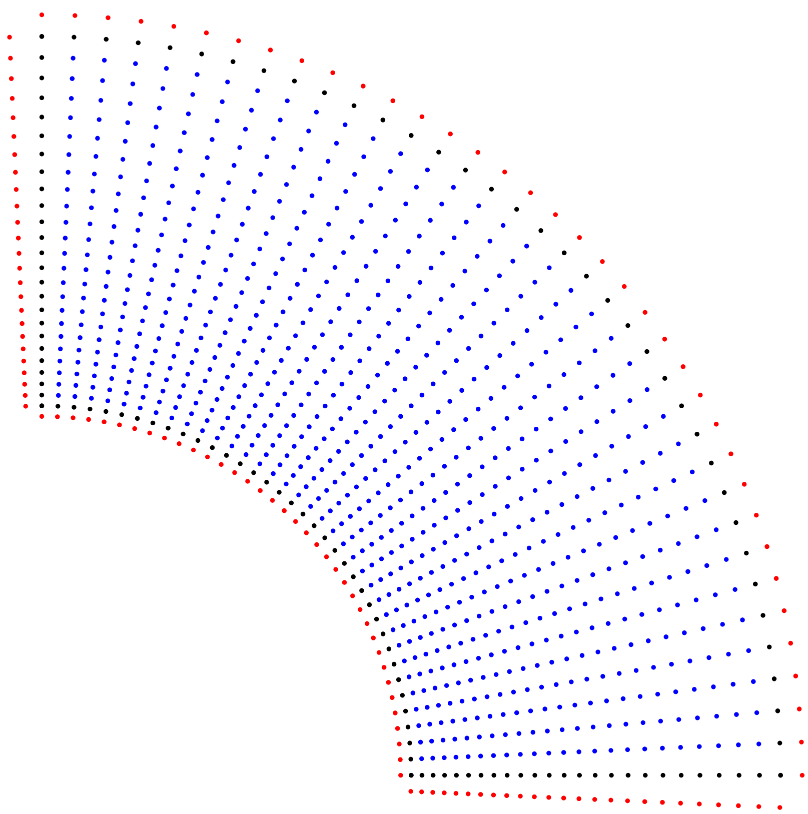

Figure 24. Computational grid used in the analysis as described in the text.

Figure 27 below shows the results for the displacement components along the outer radius of the domain. As is evident, these numerically calculated values are highly accurate.

Figures 28 and 29 below show the results for u_y and T_x along the boundary at $\theta = \pi/2$. Owing to the symmetry of the problem, these graphs look exactly like those in Figs. 25 and 26.

The final boundary results are given by Fig. 30 below, which shows the traction components T_x and T_y along the inner radius of the domain. One sees that the numerically calculated values of these traction components are highly accurate.

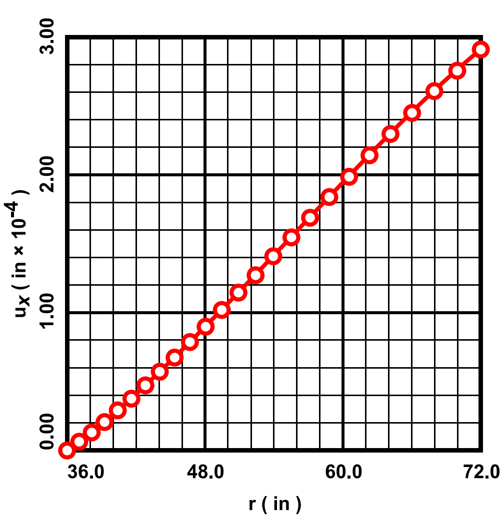

Figure 25. Analytical and numerical results for the displacement u_x along $\theta = 0$.

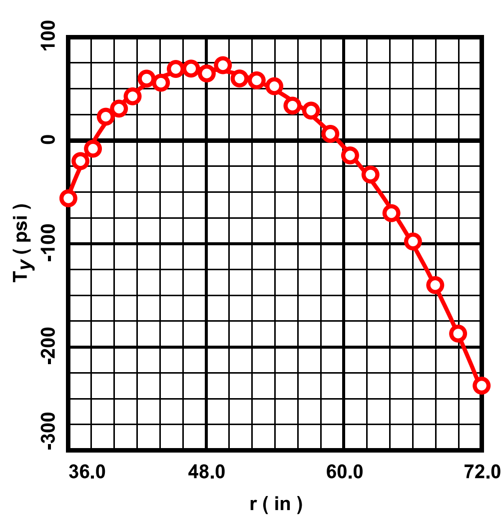

Figure 26. Analytical and numerical results for the traction T_y along $\theta = 0$.

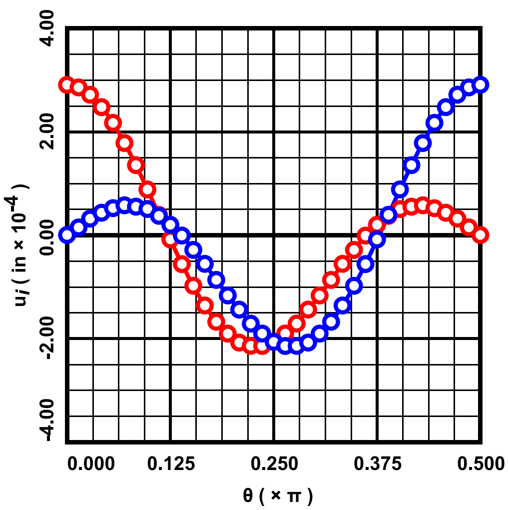

Figure 27. Analytical and numerical results for the displacements u_x (red) and u_y (blue) along $r = 72$ in.

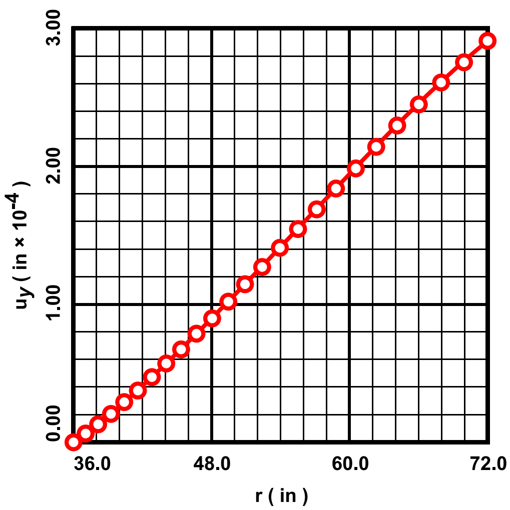

Figure 28. Analytical and numerical results for the displacement u_y along $\theta = \pi/2$.

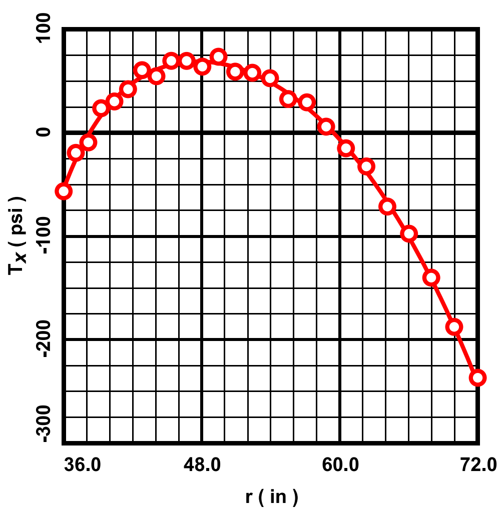

Figure 29. Analytical and numerical results for the traction T_x along $\theta = \pi/2$.

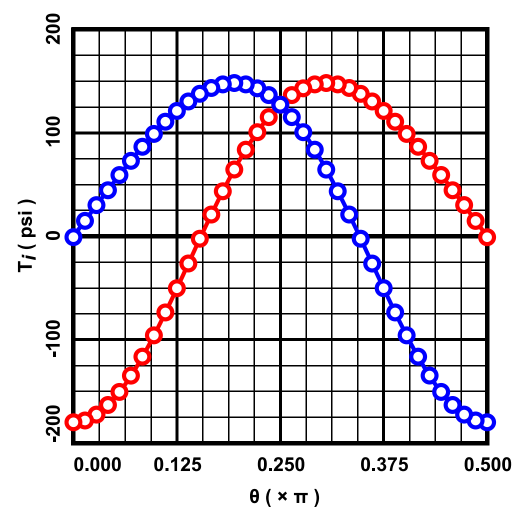

Figure 30. Analytical and numerical results for the tractions T_x (red) and T_y (blue) along $r = 36$ in.

Results of the analysis in the interior of the domain are presented below in Figs. 31 through 34. The displacement components along the radial line $\theta = \pi/8$ through the domain are shown in Fig. 31, which displacements are highly accurate. The numerically calculated values of the stress components σ_{xx} and σ_{yy} along $\theta = \pi/8$ (Fig. 32) are also highly accurate, except for the value of σ_{yy} at the outer radius (which should be zero). This probably occurs due to the finite difference procedure used to calculate the stress components on the boundary, *cf.,* Sec. 8.2. Also, results for σ_{xy} are not shown in Fig. 32. This is because, at $\theta = \pi/8$, $\sigma_{xy} = \sigma_{yy}$.

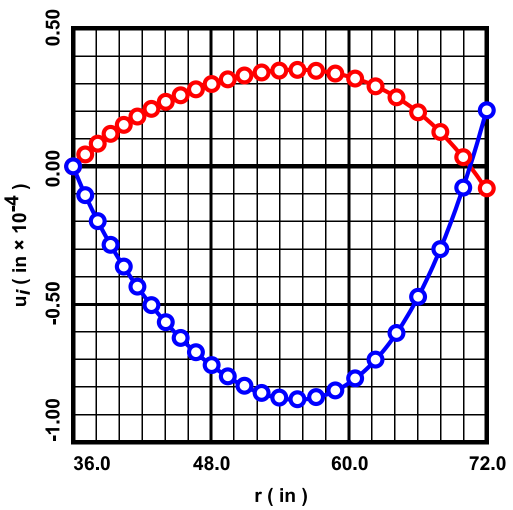

Figure 31. Analytical and numerical results for the displacements u_x (red) and u_y (blue) along $\theta = \pi/8$.

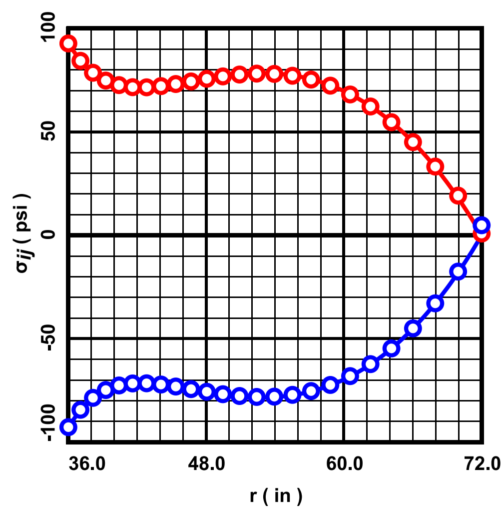

Figure 32. Analytical and numerical results for the stresses σ_{xx} (red) and σ_{yy} (blue) along $\theta = \pi/8$.

Finally, results along the radius $r = 53.9391$ in in the interior of the domain are shown below in Figs. 33 and 34. As is seen, the numerically calculated displacement components (Fig. 33) are highly accurate, as are the numerically calculated stress components (Fig. 34).

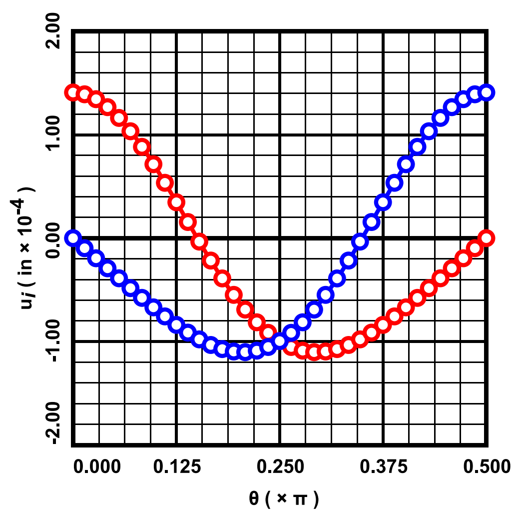

Figure 33. Analytical and numerical results for the displacements u_x (red) and u_y (blue) along $r = 53.9391$ in.

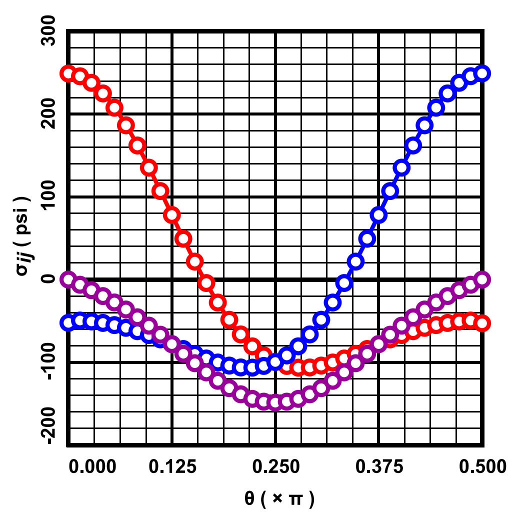

Figure 34. Analytical and numerical results for the stresses σ_{xx} (red), σ_{yy} (blue) and σ_{xy} (purple) along $r = 53.9391$ in.

18. Closing Remarks

The boundary element method presented above is both highly reliable and highly accurate (except for some minor oscillations that sometimes occur in the calculated boundary tractions). Something that the author finds curious, though, is that the method works well, even with the displacements and tractions being interpolated in the same way, *i.e.,* both linear in each element. This is counterintuitive, given that the tractions are related to the displacement gradients, so that interpolating the displacements as one order higher than the tractions would make the most sense. The author, at first, tried this approach (also using curved elements), but difficulty was encountered in obtaining non-singular systems. Nevertheless, an advantage of using linear interpolations is that it allows the integrations to be performed analytically, which yields a code that executes very quickly.